Theorem 4.7. (Lindeberg-Lévy)

If the variables of the random sequence $\{y_t\}$ are independent and have the same distribution with mean μ and variance v, then S_n converges in distribution to the standard normal distribution $N(0,1)$.

This theorem has minimal requirements for the moments of the variables but maximal requirements for their homogeneity. Note that, in this case,

$$S_n = (nv)^{-1/2} \sum_{t=1}^{n} (y_t - \mu).$$

The next theorem allows for much heterogeneity but still requires independence.

Theorem 4.8. (Lyapunov)

For each positive integer n let the finite sequence $\{y_t^n\}_{t=1}^n$ consist of independent centered random variables possessing variances v_t^n. Let $s_n^2 \equiv \sum_{t=1}^n v_t^n$ and let the **Lindeberg condition** be satisfied, namely, that for all $\varepsilon > 0$

$$\lim_{n\to\infty} \left(\sum_{t=1}^{n} s_n^{-2} E\big((y_t^n)^2 I_G(y_t^n)\big) \right) = 0,$$

where the set G used in the indicator function is $\{y : |y| \geq \varepsilon s_n\}$. Then $s_n^{-1} \sum_{t=1}^n y_t^n$ converges in distribution to $N(0,1)$.

Our last central limit theorem allows for dependent sequences.

Theorem 4.9. (McLeish)

For each positive integer n let the finite sequences $\{y_t^n\}_{t=1}^n$ be martingale difference sequences with $v_t^n \equiv \mathrm{Var}(y_t^n) < \infty$, and $s_n^2 \equiv \sum_{t=1}^n v_t^n$. If for all $\varepsilon > 0$

$$\lim_{n\to\infty} \left(s_n^{-2} \sum_{t=1}^{n} E\big((y_t^n)^2 I_G(y_t^n)\big) \right) = 0,$$

where again the set $G \equiv \{y : |y| \geq \varepsilon s_n\}$, and if the sequence

$$\left\{ \sum_{t=1}^{n} \frac{(y_t^n)^2}{s_n^2} \right\}$$

obeys a law of large numbers and thus converges to 1, then $s_n^{-1} \sum_{t=1}^n y_t^n$ converges in distribution to $N(0,1)$.

See McLeish (1974). Observe the extra condition needed in this theorem, which ensures that the variance of the limiting distribution is the same as the limit of the variances of the variables in $s_n^{-1} \sum_{t=1}^n y_t^n$.