interested in the nondegenerate asymptotic distribution of the sample mean as an estimator. We saw in Section 4.3 that for this purpose we should look at the distribution of $n^{1/2}(m_1 - \mu)$, where m_1 is the sample mean. Specifically, we wish to study

$$n^{1/2}(m_1 - \mu) = n^{-1/2} \sum_{t=1}^{n} (y_t - \mu),$$

where $y_t - \mu$ has variance σ_t^2.

We begin by stating the following simple **central limit theorem**.

Theorem 4.2. Simple Central Limit Theorem. (Lyapunov)

Let $\{y_t\}$ be a sequence of independent, centered random variables with variances σ_t^2 such that $\underline{\sigma}^2 \leq \sigma_t^2 \leq \overline{\sigma}^2$ for two finite positive constants, $\underline{\sigma}^2$ and $\overline{\sigma}^2$, and absolute third moments μ_3 such that $\mu_3 \leq \bar{\mu}_3$ for a finite constant $\bar{\mu}_3$. Further, let

$$\sigma_0^2 \equiv \lim_{n\to\infty} \left(\frac{1}{n} \sum_{t=1}^{n} \sigma_t^2 \right)$$

exist. Then the sequence

$$\left\{ n^{-1/2} \sum_{t=1}^{n} y_t \right\}$$

tends in distribution to a limit characterized by the normal distribution with mean zero and variance σ_0^2.

Theorem 4.2 applies directly to the example (4.26). Thus our hypothetical investigator may, within the limits of asymptotic theory, use the $N(0, \sigma_0^2)$ distribution for statistical inference on the estimate m_1 via the random variable $n^{1/2}(m_1 - \mu)$. Knowledge of σ_0^2 is not necessary, provided that it can be estimated consistently.

Although we do not intend to offer a formal proof of even this simple central limit theorem, in view of the technicalities that such a proof would entail, it is not difficult to give a general idea of why the result is true. For simplicity, let us consider the case in which all the variables y_t of the sequence $\{y_t\}$ have the same distribution with variance σ^2. Then clearly the variable

$$S_n \equiv n^{-1/2} \sum_{t=1}^{n} y_t$$

has mean zero and variance σ^2 for each n. But what of the higher moments of S_n? By way of an example, consider the fourth moment. It is

$$E(S_n^4) = \frac{1}{n^2} \sum_{r=1}^{n} \sum_{s=1}^{n} \sum_{t=1}^{n} \sum_{u=1}^{n} E(y_r y_s y_t y_u). \tag{4.27}$$