since $\boldsymbol{P}_1$ plays the same role for the manifold $\mathcal{R}$ as does $\boldsymbol{P}_0$ for $\mathcal{X}$. The LM statistic (3.48) is

$$\frac{1}{\tilde{\sigma}^2}(\boldsymbol{y}-\tilde{\boldsymbol{x}})^{\top}\tilde{\boldsymbol{P}}_X(\boldsymbol{y}-\tilde{\boldsymbol{x}}). \tag{5.76}$$

If we express the statistic in terms of quantities that are $O(1)$, we obtain

$$\frac{1}{\tilde{\sigma}^2}n^{-1/2}(\boldsymbol{y}-\tilde{\boldsymbol{x}})^{\top}\tilde{\boldsymbol{X}}\big(n^{-1}\tilde{\boldsymbol{X}}^{\top}\tilde{\boldsymbol{X}}\big)^{-1}n^{-1/2}\tilde{\boldsymbol{X}}^{\top}(\boldsymbol{y}-\tilde{\boldsymbol{x}}). \tag{5.77}$$

Like $\hat{\boldsymbol{X}}_t$, $\tilde{\boldsymbol{X}}_t$ is asymptotically nonstochastic. Therefore, from (5.75),

$$\begin{aligned} n^{-1/2}\tilde{\boldsymbol{X}}^{\top}(\boldsymbol{y}-\tilde{\boldsymbol{x}}) &= n^{-1/2}\sum_{t=1}^{n}\tilde{\boldsymbol{X}}_t^{\top}\tilde{u}_t \\ &= n^{-1/2}\sum_{t=1}^{n}\boldsymbol{X}_{0t}^{\top}(\boldsymbol{M}_1\boldsymbol{u})_t + o(1) \\ &= n^{-1/2}\sum_{t=1}^{n}(\boldsymbol{M}_1\boldsymbol{X}_0)_t u_t + o(1) \\ &= n^{-1/2}\boldsymbol{X}_0^{\top}\boldsymbol{M}_1\boldsymbol{u} + o(1). \end{aligned}$$

The matrix $n^{-1}\tilde{\boldsymbol{X}}^{\top}\tilde{\boldsymbol{X}}$ is asymptotically nonstochastic, just as $n^{-1}\hat{\boldsymbol{X}}^{\top}\hat{\boldsymbol{X}}$ is, and so the LM statistic (5.77) is asymptotically equivalent to

$$\boldsymbol{u}^{\top}\boldsymbol{M}_1\boldsymbol{X}_0\big(\sigma_0^2\boldsymbol{X}_0^{\top}\boldsymbol{X}_0\big)^{-1}\boldsymbol{X}_0^{\top}\boldsymbol{M}_1\boldsymbol{u} = \sigma_0^{-2}\boldsymbol{u}^{\top}\boldsymbol{M}_1\boldsymbol{P}_0\boldsymbol{M}_1\boldsymbol{u}. \tag{5.78}$$

Since $\mathcal{S}(\boldsymbol{X}_1)$ is a subspace of $\mathcal{S}(\boldsymbol{X}_0)$, we have $\boldsymbol{P}_1\boldsymbol{P}_0 = \boldsymbol{P}_0\boldsymbol{P}_1 = \boldsymbol{P}_1$, from which it follows that $\boldsymbol{M}_1\boldsymbol{P}_0\boldsymbol{M}_1 = \boldsymbol{P}_0 - \boldsymbol{P}_1$. Expression (5.78) thus becomes

$$\sigma_0^{-2}\boldsymbol{u}^{\top}(\boldsymbol{P}_0-\boldsymbol{P}_1)\boldsymbol{u} = \sigma_0^{-2}\boldsymbol{u}^{\top}\boldsymbol{P}_2\boldsymbol{u}. \tag{5.79}$$

Comparison of (5.79) with (5.72) shows that the LM statistic is asymptotically equal to the Wald statistic. Thus it too is asymptotically $\chi^2(r)$ under the null hypothesis.

The third of the three test statistics discussed in Section 3.6 was the one based on the likelihood ratio principle, the pseudo-F statistic (3.50). Since we are interested in asymptotic results only, we rewrite it here in a form in which it should be asymptotically distributed as $\chi^2(r)$:

$$\frac{1}{s^2}\big(SSR(\tilde{\boldsymbol{\beta}}) - SSR(\hat{\boldsymbol{\beta}})\big) \tag{5.80}$$

and will (somewhat loosely) refer to it as the LR statistic. We have already seen that $s^2 \rightarrow \sigma_0^2$ as $n \rightarrow \infty$. It remains to show that $SSR(\tilde{\boldsymbol{\beta}}) - SSR(\hat{\boldsymbol{\beta}})$, when divided by σ_0^2, is asymptotically $\chi^2(r)$. From (5.64), we have

$$\hat{\sigma}^2 = \frac{1}{n}\boldsymbol{u}^{\top}\boldsymbol{M}_0\boldsymbol{u} + o(n^{-1}),$$