The first term in the second line is

$$\begin{aligned}
&\frac{1}{n}\boldsymbol{u}^\top(\boldsymbol{M}_0^\Omega)^\top\boldsymbol{\Omega}^{-1}\boldsymbol{M}_0^\Omega\boldsymbol{u} \\
&= \frac{1}{n}\boldsymbol{u}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{u} - \frac{2}{n}\boldsymbol{u}^\top(\boldsymbol{P}_0^\Omega)^\top\boldsymbol{\Omega}^{-1}\boldsymbol{u} + \frac{1}{n}\boldsymbol{u}^\top(\boldsymbol{P}_0^\Omega)^\top\boldsymbol{\Omega}^{-1}\boldsymbol{P}_0^\Omega\boldsymbol{u} \\
&= \frac{1}{n}\boldsymbol{u}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{u} - \frac{1}{n}\boldsymbol{u}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{P}_0^\Omega\boldsymbol{u},
\end{aligned} \tag{9.19}$$

where

$$\boldsymbol{P}_0^\Omega \equiv \mathbf{I} - \boldsymbol{M}_0^\Omega \equiv \boldsymbol{X}_0\big(\boldsymbol{X}_0^\top\boldsymbol{\Omega}^{-1}\boldsymbol{X}_0\big)^{-1}\boldsymbol{X}_0^\top\boldsymbol{\Omega}^{-1}$$

is essentially the same as $\boldsymbol{P}_X^\Omega$ defined in (9.12). Only the first term of (9.19) is $O(1)$. Intuitively, the reason for this is that when $\boldsymbol{u}$ is projected onto $\mathcal{S}(\boldsymbol{X}_0)$, the result lies in a k–dimensional space. Thus an expression like the second term of (9.19), which can be written as

$$n^{-1}\big(n^{-1/2}\boldsymbol{u}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{X}_0\big)\big(n^{-1}\boldsymbol{X}_0^\top\boldsymbol{\Omega}^{-1}\boldsymbol{X}_0\big)^{-1}\big(n^{-1/2}\boldsymbol{X}_0^\top\boldsymbol{\Omega}^{-1}\boldsymbol{u}\big),$$

is $O(n^{-1})$, since every factor except the first is $O(1)$.

From (9.18) and (9.19) we conclude that

$$\frac{1}{n}\tilde{\boldsymbol{u}}^\top\boldsymbol{\Omega}^{-1}\tilde{\boldsymbol{u}} \overset{a}{=} \frac{1}{n}\boldsymbol{u}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{u}. \tag{9.20}$$

The quadratic form on the right-hand side of (9.20) can be expressed very simply by using a matrix $\boldsymbol{\eta}$ that satisfies (9.08). We obtain

$$\frac{1}{n}\boldsymbol{u}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{u} = \frac{1}{n}\sum_{t=1}^{n}(\boldsymbol{\eta}\boldsymbol{u})_t^2.$$

The vector $\boldsymbol{\eta}\boldsymbol{u}$ has mean zero and variance matrix equal to $\mathbf{I}_n$. The terms of the sum of the right-hand side of this expression are therefore uncorrelated and asymptotically independent. Thus we may apply a law of large numbers and assert that the probability limit of the sum is unity. It follows that

$$\operatorname*{plim}_{n\to\infty}\Big(\frac{1}{n}\boldsymbol{u}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{u}\Big) = 1.$$

From (9.20), we then conclude that this is still true if $\boldsymbol{u}$ is replaced by $\tilde{\boldsymbol{u}}$, which was what we originally set out to show.

This result can be used to test whether $\boldsymbol{\Omega}$ really is the covariance matrix of the error terms. An appropriate test statistic is $\tilde{\boldsymbol{u}}^\top\boldsymbol{\Omega}^{-1}\tilde{\boldsymbol{u}}$, which is simply the SSR from the original GNLS regression after transformation. It should be asymptotically distributed as $\chi^2(n-k)$ under the null hypothesis.