For a regression model like (11.27), it is easy to compute a DWH test by means of an artificial regression. We saw some examples of this in Section 7.9 and will discuss further examples below. However, there is another way to compute DWH tests, and it can be more convenient in some cases. For some model that need not necessarily be a regression model, let $\hat{\boldsymbol{\theta}}$ denote an efficient estimator of the model parameters and $\check{\boldsymbol{\theta}}$ an estimator that is less efficient but consistent under weaker conditions than those of the model. Let us denote the vector of contrasts between $\check{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\theta}}$ by $\boldsymbol{e}$. Then we have seen that

$$n^{1/2}(\check{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \stackrel{a}{=} n^{1/2}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) + n^{1/2}\boldsymbol{e}, \tag{11.30}$$

where $n^{1/2}\boldsymbol{e}$ is asymptotically uncorrelated with $n^{1/2}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)$. This result was proved for models estimated by maximum likelihood in Section 8.8; its finite-sample equivalent for linear regression models was proved as part of the proof of the Gauss-Markov Theorem in Section 5.5. Because the two terms on the right-hand side of (11.30) are asymptotically uncorrelated, the asymptotic covariance matrix of the left-hand side is just the sum of the asymptotic covariance matrices of those two terms. Therefore, we obtain

$$\lim_{n\to\infty} \boldsymbol{V}\big(n^{1/2}(\check{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)\big) = \lim_{n\to\infty} \boldsymbol{V}\big(n^{1/2}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0)\big) + \lim_{n\to\infty} \boldsymbol{V}(n^{1/2}\boldsymbol{e}),$$

from which, in simplified notation, we may deduce the asymptotic covariance matrix of the vector of contrasts:

$$\boldsymbol{V}^\infty(\check{\boldsymbol{\theta}} - \hat{\boldsymbol{\theta}}) = \boldsymbol{V}^\infty(\check{\boldsymbol{\theta}}) - \boldsymbol{V}^\infty(\hat{\boldsymbol{\theta}}). \tag{11.31}$$

In words, the asymptotic covariance matrix of the difference between $\check{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\theta}}$ is equal to the difference of their respective asymptotic covariance matrices. This important result is due to Hausman (1978).

The result (11.31) can be used to construct DWH tests of the form

$$(\check{\boldsymbol{\theta}} - \hat{\boldsymbol{\theta}})^\top \big(\check{\boldsymbol{V}}(\check{\boldsymbol{\theta}}) - \hat{\boldsymbol{V}}(\hat{\boldsymbol{\theta}})\big)^{-1} (\check{\boldsymbol{\theta}} - \hat{\boldsymbol{\theta}}), \tag{11.32}$$

where $\check{\boldsymbol{V}}(\check{\boldsymbol{\theta}})$ and $\hat{\boldsymbol{V}}(\hat{\boldsymbol{\theta}})$ denote estimates of the covariance matrices of $\check{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\theta}}$, respectively. The test statistic (11.32) will be asymptotically distributed as chi-squared with as many degrees of freedom as the rank of $\boldsymbol{V}^\infty(\check{\boldsymbol{\theta}}) - \boldsymbol{V}^\infty(\hat{\boldsymbol{\theta}})$. Note that the inverse in (11.32) will have to be replaced by a generalized inverse if, as is often the case, the rank of $\boldsymbol{V}^\infty(\check{\boldsymbol{\theta}}) - \boldsymbol{V}^\infty(\hat{\boldsymbol{\theta}})$ is less than the number of parameters in $\boldsymbol{\theta}$; see Hausman and Taylor (1982). There can be practical difficulties with (11.32) if $\check{\boldsymbol{V}}(\check{\boldsymbol{\theta}}) - \hat{\boldsymbol{V}}(\hat{\boldsymbol{\theta}})$ is not positive semidefinite or if the rank of $\check{\boldsymbol{V}}(\check{\boldsymbol{\theta}}) - \hat{\boldsymbol{V}}(\hat{\boldsymbol{\theta}})$ differs from the rank of $\boldsymbol{V}^\infty(\check{\boldsymbol{\theta}}) - \boldsymbol{V}^\infty(\hat{\boldsymbol{\theta}})$. That is why we emphasize the approach based on artificial regressions.

In the case of the linear regression (11.27), where the two estimators are (11.28) and (11.29), the DWH test is based on the vector of contrasts

$$\check{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}} = \big(\boldsymbol{X}^\top\boldsymbol{A}\boldsymbol{X}\big)^{-1}\boldsymbol{X}^\top\boldsymbol{A}\boldsymbol{M}_X\boldsymbol{y}. \tag{11.33}$$