condition. Unlike asymptotic equality, the big-$O$ relation does not require that the ratio $f(n)/g(n)$ should have any limit. It may have, but it may also oscillate boundedly for ever.

The relations we have defined so far are for nonstochastic real-valued sequences. Of greater interest to econometricians are the so-called **stochastic order relations**. These are perfectly analogous to the relations we have defined but instead use one or other of the forms of stochastic convergence. Formally:

*Definition 4.8.*

If $\{a_n\}$ is a sequence of random variables, and $g(n)$ is a real-valued function of the positive integer argument $n$, then the notation $a_n = o_p\big(g(n)\big)$ means that

$$\operatorname*{plim}_{n\to\infty}\left(\frac{a_n}{g(n)}\right) = 0.$$

Similarly, the notation $a_n = O_p\big(g(n)\big)$ means that, for all $\varepsilon > 0$, there exist a constant $K$ and a positive integer $N$ such that

$$\Pr\left(\left|\frac{a_n}{g(n)}\right| > K\right) < \varepsilon \quad \text{for all } n > N.$$

If $\{b_n\}$ is another sequence of random variables, the notation $a_n \overset{a}{=} b_n$ means that

$$\operatorname*{plim}_{n\to\infty}\left(\frac{a_n}{b_n}\right) = 1.$$

Comparable definitions may be written down for almost sure convergence and convergence in distribution, but we will not use these. In fact, after this section we will not bother to use the subscript $p$ in the stochastic order symbols, because it will always be plain when random variables are involved. When they are, $O(\cdot)$ and $o(\cdot)$ should be read as $O_p(\cdot)$ and $o_p(\cdot)$.

The order symbols are very easy to manipulate, and we now present a few useful rules for doing so. For simplicity, we restrict ourselves to functions $g(n)$ that are just powers of $n$, for that is all we use in this book. The rules for addition and subtraction are

$$\begin{aligned}
O(n^p) \pm O(n^q) &= O\big(n^{\max(p,q)}\big);\\
o(n^p) \pm o(n^q) &= o\big(n^{\max(p,q)}\big);\\
O(n^p) \pm o(n^q) &= O(n^p) \quad \text{if } p \geq q;\\
O(n^p) \pm o(n^q) &= o(n^q) \quad \text{if } p < q.
\end{aligned}$$

The rules for multiplication, and by implication for division, are

$$\begin{aligned}
O(n^p)O(n^q) &= O(n^{p+q});\\
o(n^p)o(n^q) &= o(n^{p+q});\\
O(n^p)o(n^q) &= o(n^{p+q}).
\end{aligned}$$