Tutorial #1 (Sketch of Answers)

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On Parallelism of Indifference Lines (The Easy Way)

Note that with only three finite states, the expected utility theorem guarantees the existence of a utility function of additive form :

$$U(L) \equiv \sum_{i=1}^{3} p_i u_i = p_1 u_1 + p_2 u_2 + p_3 u_3$$

where u_i is the utility of having state i. Now since we talk about probabilities, we must have that $(1 - p_1 - p_2) = p_3$. Thus, we have :

$$\begin{aligned} U(L) &= p_1 u_1 + p_2 u_2 + (1 - p_2 - p_1) u_3 \\ &= p_1 (u_1 - u_3) + p_2 (u_2 - u_3) + u_3. \end{aligned}$$

Since we seek a characterization of indifference sets, we fix a utility level $\bar{U}$ and find that :

$$\begin{aligned} \bar{U} &= p_1 (u_1 - u_3) + p_2 (u_2 - u_3) + u_3 \\ \Leftrightarrow p_2 &= p_1 \underbrace{\frac{u_1 - u_3}{u_3 - u_2}}_{\equiv b} + \underbrace{\frac{u_3 - \bar{U}}{u_3 - u_2}}_{\equiv a} \end{aligned}$$

Hence, the set of lotteries with a given level of utility $\bar{U}$ is such that $p_1 = a + b p_2$ (and of course $\sum_i p_1 = 1$). You can see that the level of utility only changes the value of a. Hence, every set of indifference lines are parallels.

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On Parallelism of Indifference Lines (The Hard Way)

Claim 1 (Parallelism)**.** *Let $(X, \succeq)$ be a preference space satisfying completeness, transitivity, continuity and independance. Define $a \sim b$ if $a \succeq b$ and $b \succeq a$ and $a \succ b$ if $a \succeq b$ and $b \not\succeq a$.*

Let $\{\lambda, \mu, \lambda', \mu', v\} \subset X$ such that

$$\mu \sim \lambda \succ \mu' \sim \lambda' \succ v$$

Then, indifference sets are parallel lines.

Proof. The result is articulated in three steps. See figure 1 for the intuition. The first step is to show that there exists a p^* such that $\lambda' \sim p^*\lambda + (1-p^*)v$. The second step is to show that it is the same p^* that makes $\mu' \sim p^*\mu + (1-p^*)v$. Finally, the last step is to show that indifference sets are straight lines.

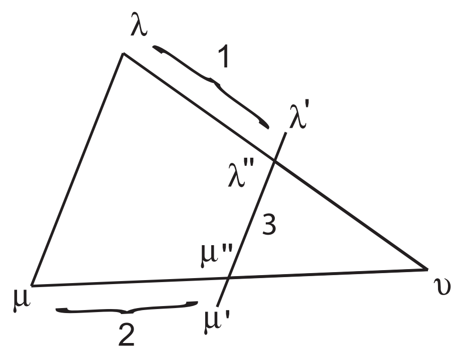

FIGURE 1 – **Strategy of proof.**

First step : By the axiom of independence, we have that

$$\lambda \succ \lambda' \Leftrightarrow p\lambda + (1-p)v \succ p\lambda' + (1-p)v \;\forall p \in [0,1]$$

By the axiom of completeness, we have that

$$p\lambda + (1-p)v \succeq \lambda' \text{ or } p\lambda + (1-p)v \preceq \lambda' \;\forall p \in [0,1]$$

(or both at the same time). Now consider the following sequence :

1. Start with $p_{high} = 1, p_{low} = 0, p_0 = 1/2$.
2. Let $T(p_i)$ be the following operation on p.
   (a) If p_{i-1} is such that $p_{i-1}\lambda + (1-p_{i-1})v \succ \lambda'$, set $p_{max} = p_{i-1}, p_i = \frac{p_{i-1}+p_{low}}{2}$.

(b) If p_{i-1} is such that $p_{i-1}\lambda + (1-p_{i-1})v \prec \lambda'$, set $p_{low} = p_{i-1}, p_i = \frac{p_{i-1}+p_{max}}{2}$.

(c) If p_{i-1} is such that $p_{i-1}\lambda + (1-p_{i-1})v \sim \lambda'$, set $p_i = p_{i-1}$.

Then, notice that p_{max} is a decreasing sequence and p_{low} is an increasing sequence. Hence, we conclude that $\{p_i\}$ is a Cauchy sequence and since $[0,1]$ is a complete space, there exist a limit p^* for which $p^* = T(p^*)$. Hence $p^*\lambda + (1-p^*)v \sim \lambda'$. Call this point λ''. With the same logic, there exists a $q^* \in [0,1]$ such that $q^*\mu + (1-q^*)v \sim \mu'$. Call this point μ''.

Now, the second step is to show that $q^* = p^*$. From transitivity, we have :

$$\begin{aligned} p^*\lambda + (1-p^*)v &\sim \lambda' \\ \lambda' &\sim \mu' \\ q^*\mu + (1-q^*)v &\sim \mu' \\ \Rightarrow q^*\mu + (1-q^*)v &\sim p^*\lambda + (1-p^*)v \end{aligned}$$

From the axiom of independance and the fact that $\lambda \succ v$, we have :

$$p\lambda + (1-p)v \succ v$$

Applying this idea again with some $\gamma \in [0,1]$ we find :

$$p\lambda + (1-p)v \succ \gamma\,[\,p\lambda + (1-p)v] + (1-\gamma)v \succ v$$

and thus if $p > q$, we can find a γ^* such that $\gamma^* p = q$. If $p < q$, note that such γ would not be in $[0,1]$. This allows us to deduce that if $p > q$,

$$p\lambda + (1-p)v \succ q\lambda + (1-q) + (1-\gamma)v \succ v$$

Coversely, if $q > p$, we get the opposite result. But now, we are almots done. By the axiom of independence and $\mu \sim \lambda$, we deduce that :

$$\begin{aligned} q^*\mu + (1-q^*)v &\sim p^*\mu + (1-p^*)v \\ q^*\lambda + (1-q^*)v &\sim p^*\lambda + (1-p^*)v \end{aligned}$$

which is possible if and only if $p^* = q^*$.

Now, the third and last step. Define the set of indifference points by

$$C_{\mu,\sim} := \{x \in X|\ x \sim \mu\}$$

By the axiom of independance, we have that :

$$\begin{aligned} \mu \succeq p\mu + (1-p)\lambda \succeq \lambda \ \forall\ p \in [0,1] \\ \lambda \succeq p\mu + (1-p)\lambda \succeq \mu \ \forall\ p \in [0,1] \end{aligned}$$

and by hypothesis, we have $\mu \sim \lambda$. From transitivity, we then deduce that $p\mu + (1-p)\lambda \in C_{\mu,\sim}$ $\forall p \in [0,1]$ and likewise for μ', λ' with $C_{\mu',\sim}$. Hence the $\ll$ line $\gg$ is in the indifference set. It remains to show that these are the only points that lies in there. Let $x \in X$ be such that $\lambda \succ x \succ v$ and such that $x \neq r\mu' + (1-r)\lambda'$ $\forall r \in [0,1]$. By the first step of this proof, we know that there exists a s^* such that $s^*\lambda + (1-s^*)\lambda \sim x$. By hypothesis, $s^* \neq p^*$. Hence, we either have $s^* > p^*, s^* < p^*$ and from step two, we deduce that it is not in $C_{\mu',\sim}$. Hence, the only points in $C_{\mu',\sim}$ are straight lines and since $p^* = q^*$, one must deduce these are parallel lines. $\square$