ECON 452* -- NOTE 15

Marginal Effects in Probit Models: Interpretation and Testing

This note introduces you to the two types of marginal effects in probit models: **marginal *index* effects**, and **marginal *probability* effects**. It demonstrates how to calculate these effects for both continuous and categorical explanatory variables.

1. Interpreting Probit Coefficients

A Generic Probit Model

- The conventional formulation of a binary dependent variable model assumes that an ***unobserved*** **(or *latent*) dependent variable** Y_i^* is generated by a classical linear regression model of the form

$$Y_i^* = x_i^T\beta + u_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \cdots + \beta_k X_{ik} + u_i \qquad (1)$$

where:

Y_i^* = a continuous real-valued index variable for observation i that is *unobservable*, or *latent*;

$x_i^T = (1 \; X_{i1} \; X_{i2} \; \cdots \; X_{ik})$, a 1×K row vector of regressor values for observation i;

$\beta = (\beta_0 \; \beta_1 \; \beta_2 \; \cdots \; \beta_k)^T$, a K×1 column vector of regression coefficients;

$x_i^T\beta$ = a 1×1 scalar called the ***index function*** for observation i;

u_i = an iid $N(0, \sigma^2)$ random error term for observation i.

- The ***observable outcomes*** **of the binary choice problem** are represented by a **binary indicator variable Y_i** that is related to the unobserved dependent variable Y_i^* as follows:

$$Y_i = 1 \text{ if } Y_i^* > 0 \tag{2.1}$$

$$Y_i = 0 \text{ if } Y_i^* \le 0 \tag{2.2}$$

- The **random indicator variable Y_i** represents the observed realizations of a binomial process with the following probabilities:

$$\Pr(Y_i = 1) = \Pr(Y_i^* > 0) = \Pr(x_i^T\beta + u_i > 0) \tag{3.1}$$

$$\Pr(Y_i = 0) = \Pr(Y_i^* \le 0) = \Pr(x_i^T\beta + u_i \le 0) \tag{3.2}$$

- **Probit models** analytically represent the binomial probabilities (3.1) and (3.2) in terms of the standard normal c.d.f. $\Phi(Z)$ as follows:

$$\Pr(Y_i = 1) = \Pr(Y_i^* > 0) = \Phi\left(x_i^T\beta\right) \tag{4.1}$$

$$\Pr(Y_i = 0) = \Pr(Y_i^* \le 0) = 1 - \Phi\left(x_i^T\beta\right) \tag{4.2}$$

- **Interpretation of the probit coefficient vector β**

♦ Under the zero conditional mean error assumption, equation (1) implies that

$$E\left(Y_i^*\middle|\, x_i^T\right) = E\left(x_i^T\beta\middle|\, x_i^T\right) + E\left(u_i\middle|\, x_i^T\right) = x_i^T\beta \quad \text{since } E\left(u_i\middle|\, x_i^T\right) = 0. \tag{5}$$

♦ The ***index function (or regression function)*** $x_i^T\beta$ is thus the conditional mean value of the latent random variable Y_i^* for given values of the regressors.

♦ The ***slope coefficients*** $\boldsymbol{\beta_j}$ (j = 1, …, k): If all explanatory variables are *continuous* and enter the index function *linearly*, the **partial derivatives of regression function (5)** with respect to the individual regressors are the ***slope coefficients*** $\boldsymbol{\beta_j}$ (j = 1, …, k):

$$\frac{\partial E\left(Y_i^*\middle|\, x_i^T\right)}{\partial X_{ij}} = \frac{\partial x_i^T\beta}{\partial X_{ij}} = \frac{\partial(\beta_0 + \beta_1 X_{i1} + \cdots + \beta_j X_{ij} + \cdots + \beta_k X_{ik})}{\partial X_{ij}} = \beta_j.$$

♦ But if some of the explanatory variables are *binary* or enter the index function *nonlinearly*, the **partial derivatives of regression function (5)** are not so simply interpreted.

2. Two Types of Marginal Effects in Probit Models

For each explanatory variable, there are two types of marginal effects in binary dependent variables models.

<u>Marginal Index Effects</u>

Marginal *index* effects are the partial effects of each explanatory variable on the **probit index function** $x_i^T\beta$.

- ***<u>Case 1</u>:*** **X_j is a *continuous* explanatory variable**

  marginal *index* effect of variable X_j $\equiv \dfrac{\partial E\left(Y_i^* \middle| x_i^T\right)}{\partial X_{ij}} = \dfrac{\partial x_i^T\beta}{\partial X_{ij}}$

- ***<u>Case 2</u>:*** **X_j is a *binary* explanatory variable** (a ***dummy*** **or** ***indicator*** **variable**)

  The marginal index effect of a binary explanatory variable equals

  1. the value of the index function $x_i^T\beta$ **when $X_{ij} = 1$** and the other regressors equal specified fixed values

  minus

  2. the value of the index function $x_i^T\beta$ **when $X_{ij} = 0$** and the other regressors equal the *same* fixed values

- ***Case 2:*** **X_j is a *binary* explanatory variable** (a ***dummy*** **or** ***indicator*** **variable**)

Formal Definition: Define two different vectors of regressor values in which all explanatory variables except X_{ij} are held constant at fixed values:

x_{1i}^T = any vector of regressor values with $X_{ij} = 1$ (and all other explanatory variables equal to fixed values);
x_{0i}^T = the same vector of regressor values but with $X_{ij} = 0$.

The **marginal *index* effect of the *binary* (*dummy*) variable X_j** is:

$$\textbf{marginal } \textbf{\textit{index}} \textbf{ effect of } \mathbf{X_j} = E\left(Y_i^* \mid X_{ij}=1, X_{ih}=X_{0h}\ \forall\ h \neq j\right) - E\left(Y_i^* \mid X_{ij}=0, X_{ih}=X_{0h}\ \forall\ h \neq j\right)$$

$$= x_{1i}^T\beta - x_{0i}^T\beta$$

Limitation: Marginal index effects are difficult to interpret because it is difficult to interpret – and impossible to measure – the latent dependent variable Y_i^*.

<u>Marginal Probability Effects</u>

Marginal *probability* effects are the partial effects of each explanatory variable on the **probability that the observed dependent variable $Y_i = 1$,** where in probit models

$\Pr(Y_i = 1) = \Phi(x_i^T\beta)$ = standard normal c.d.f. evaluated at $x_i^T\beta$.

- ***<u>Case 1</u>:* X_j is a *continuous* explanatory variable**

marginal *probability* effect of variable X_j $\equiv \dfrac{\partial \Pr(Y_i = 1)}{\partial X_{ij}} = \dfrac{\partial \Phi(x_i^T\beta)}{\partial X_{ij}}$

Using the **chain rule of differentiation**, we can obtain a general expression for the marginal probability effect of a continuous explanatory variable X_j:

marginal *probability* effect of X_j

$$= \frac{\partial \Phi(x_i^T\beta)}{\partial X_{ij}} = \frac{d\,\Phi(x_i^T\beta)}{d\,(x_i^T\beta)}\frac{\partial x_i^T\beta}{\partial X_{ij}} = \phi(x_i^T\beta)\frac{\partial x_i^T\beta}{\partial X_{ij}}$$

where

$\phi(x_i^T\beta) = \dfrac{d\,\Phi(x_i^T\beta)}{d\,(x_i^T\beta)} =$ the value of the standard normal p.d.f. at $x_i^T\beta$.

$\dfrac{\partial x_i^T\beta}{\partial X_{ij}} =$ the **marginal *index* effect of X_j**

- ***<u>Case 2</u>:*** **X_j is a *binary* explanatory variable** (a ***dummy*** **or** ***indicator*** **variable**)

The **marginal *probability* effect of a *binary* explanatory variable** equals

1. the value of $\Phi(x_i^T\beta)$ when X_{ij} = 1 and the other explanatory variables X_{ih} ($h \neq j$) equal the fixed values X_{0h}

minus

2. value of $\Phi(x_i^T\beta)$ when X_{ij} = 0 and the other explanatory variables X_{ih} ($h \neq j$) equal the *same* fixed values X_{0h}

Formal Definition: Define two different vectors of regressor values:

x_{1i}^T = any vector of regressor values with $X_{ij} = 1$;
x_{0i}^T = the same vector of regressor values but with $X_{ij} = 0$.

The **marginal *probability* effect of the *binary* (*dummy*) variable X_j** is:

$$\textbf{marginal } \textbf{\textit{probability}} \textbf{ effect of } \mathbf{X_j} = \Pr\left(Y_i = 1 \,\middle|\, X_{ij} = 1, X_{ih} = X_{0h} \;\forall\, h \neq j\right) - \Pr\left(Y_i = 1 \,\middle|\, X_{ij} = 0, X_{ih} = X_{0h} \;\forall\, h \neq j\right)$$

$$= \Phi\left(x_{1i}^T\beta\right) - \Phi\left(x_{0i}^T\beta\right).$$

Relationship Between the Two Marginal Effects for Continuous Variables

- Compare the marginal index effect and marginal probability effect of a ***continuous*** **explanatory variable X_j**.

  marginal *index* effect of variable X_j $= \dfrac{\partial x_i^T\beta}{\partial X_{ij}}$

  marginal *probability* effect of variable X_j $= \dfrac{\partial \Phi(x_i^T\beta)}{\partial X_{ij}} = \phi(x_i^T\beta)\dfrac{\partial x_i^T\beta}{\partial X_{ij}}$

- ***Relationship:*** For a continuous explanatory variable X_j, the marginal probability effect is proportional to the marginal index effect of X_j, where the factor of proportionality is the standard normal p.d.f. at $x_i^T\beta$:

  marginal *probability* effect of X_j $= \phi(x_i^T\beta) \times$ **marginal *index* effect of X_j**

3. Marginal Index and Probability Effects in Probit Models

A Simple Probit Model

$$Y_i^* = x_i^T\beta + u_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i2}^2 + \beta_4 X_{i3} + \beta_5 D_i + \beta_6 D_i X_{i3} + u_i$$

where:

X_{i1}, X_{i2} and X_{i3} are *continuous* explanatory variables

D_i is a *binary* (or *dummy*) explanatory variable defined such that
D_i = 1 if observation i exhibits some attribute, = 0 otherwise.

- The **index function** is:

$$x_i^T\beta = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i2}^2 + \beta_4 X_{i3} + \beta_5 D_i + \beta_6 D_i X_{i3}$$

  - X_{i1} enters the index function *linearly*.
  - X_{i2} enters the index function *nonlinearly*.
  - X_{i3} enters the index function *nonlinearly*, interacted with D_i.
  - D_i enters the index function *nonlinearly*, interacted with X_{i3}.

- The ***observed*** **binary dependent variable Y_i** is related to the unobserved dependent variable Y_i^* as follows:

$$Y_i = 1 \text{ if } Y_i^* > 0$$
$$Y_i = 0 \text{ if } Y_i^* \le 0$$

- The **binomial probabilities** $\Pr(Y_i = 1)$ and $\Pr(Y_i = 0)$ are analytically represented in probit models in terms of the standard normal c.d.f. $\Phi(Z)$:

$$\begin{aligned}\Pr(Y_i = 1) &= \Pr(Y_i^* > 0) = \Phi\left(x_i^T\beta\right) \\ &= \Phi\left(\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i2}^2 + \beta_4 X_{i3} + \beta_5 D_i + \beta_6 D_i X_{i3}\right)\end{aligned}$$

$$\begin{aligned}\Pr(Y_i = 0) &= \Pr(Y_i^* \le 0) = 1 - \Phi\left(x_i^T\beta\right) \\ &= 1 - \Phi\left(\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i2}^2 + \beta_4 X_{i3} + \beta_5 D_i + \beta_6 D_i X_{i3}\right)\end{aligned}$$

Marginal Effects of X_1* = a *continuous* variable that enters *linearly

$$x_i^T\beta = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i2}^2 + \beta_4 X_{i3} + \beta_5 D_i + \beta_6 D_i X_{i3}$$

- **Marginal *index* effect of *X_1***

marginal index effect of X_1 = $\dfrac{\partial x_i^T\beta}{\partial X_{i1}} = \beta_1$

- **Marginal *probability* effect of *X_1***

marginal probability effect of X_1 = $\phi\left(x_i^T\beta\right)\dfrac{\partial x_i^T\beta}{\partial X_{i1}} = \phi\left(x_i^T\beta\right)\beta_1$

Marginal Effects of X_2* = a *continuous* variable that enters *nonlinearly

- **Marginal *index* effect of *X_2***

marginal index effect of X_2 = $\dfrac{\partial x_i^T\beta}{\partial X_{i2}} = \beta_2 + 2\beta_3 X_{i2}$

- **Marginal *probability* effect of *X_2***

marginal probability effect of X_2 = $\phi\left(x_i^T\beta\right)\dfrac{\partial x_i^T\beta}{\partial X_{i2}} = \phi\left(x_i^T\beta\right)\left(\beta_2 + 2\beta_3 X_{i2}\right)$

Marginal Effects of X_3* = a *continuous* variable that enters *nonlinearly

$$x_i^T\beta = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i2}^2 + \beta_4 X_{i3} + \beta_5 D_i + \beta_6 D_i X_{i3}$$

- **Marginal *index* effect of *X_3***

marginal index effect of X_3 = $\frac{\partial x_i^T\beta}{\partial X_{i3}} = \beta_4 + \beta_6 D_i = \beta_4 + \beta_6$ when $D_i = 1$

$= \beta_4$ when $D_i = 0$

- **Marginal *probability* effect of *X_3***

marginal probability effect of X_3 = $\phi(x_i^T\beta)\frac{\partial x_i^T\beta}{\partial X_{i3}} = \phi(x_i^T\beta)(\beta_4 + \beta_6 D_i)$

$= \phi(x_i^T\beta)(\beta_4 + \beta_6)$ when $D_i = 1$

$= \phi(x_i^T\beta)\,\beta_4$ when $D_i = 0$

Marginal Effects of D* = a *binary* variable that enters *nonlinearly

The **regressor vector for sample observation i** is: $x_i^T = \begin{pmatrix} 1 & X_{i1} & X_{i2} & X_{i2}^2 & X_{i3} & D_i & D_iX_{i3} \end{pmatrix}$

- **Marginal *index* effect of D**

  Define two vectors of regressor values that contain the *same values* X_{i1}, X_{i2} and X_{i3} of the other three explanatory variables X_1, X_2 and X_3:

  one with $D_i = 1$: $x_{1i}^T = \begin{pmatrix} 1 & X_{i1} & X_{i2} & X_{i2}^2 & X_{i3} & 1 & X_{i3} \end{pmatrix}$

  the other with $D_i = 0$: $x_{0i}^T = \begin{pmatrix} 1 & X_{i1} & X_{i2} & X_{i2}^2 & X_{i3} & 0 & 0 \end{pmatrix}$

  The corresponding **values of the index function** are:

  for $D_i = 1$:
  $$\begin{aligned} x_{1i}^T\beta &= \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i2}^2 + \beta_4 X_{i3} + \beta_5 + \beta_6 X_{i3} \\ &= \beta_0 + \beta_5 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i2}^2 + (\beta_4 + \beta_6) X_{i3} \end{aligned}$$

  for $D_i = 0$:
  $$x_{0i}^T\beta = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i2}^2 + \beta_4 X_{i3}$$

  The **index function difference** $= x_{1i}^T\beta - x_{0i}^T\beta = \beta_5 + \beta_6 X_{i3}$

  The **marginal *index* effect of D** is therefore:

  marginal index effect of D = $x_{1i}^T\beta - x_{0i}^T\beta = \beta_5 + \beta_6 X_{i3}$

$$x_i^T\beta = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i2}^2 + \beta_4 X_{i3} + \beta_5 D_i + \beta_6 D_i X_{i3}$$

Recall that the probit index functions for $D_i = 1$ and $D_i = 0$ are given respectively by:

$$x_{1i}^T\beta = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i2}^2 + \beta_4 X_{i3} + \beta_5 + \beta_6 X_{i3} = \beta_0 + \beta_5 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i2}^2 + (\beta_4 + \beta_6) X_{i3}$$

$$x_{0i}^T\beta = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i2}^2 + \beta_4 X_{i3}$$

- **Marginal *probability* effect of *binary* dummy variable D** with the other three explanatory variables X_1, X_2, and X_3 equal respectively to the fixed values X_{01}, X_{02}, and X_{03} is:

$$= \Pr\left(Y_i = 1 \mid D_i = 1, X_{ih} = X_{0h}\ \forall\ h = 1, 2, 3\right) - \Pr\left(Y_i = 1 \mid D_i = 0, X_{ih} = X_{0h}\ \forall\ h = 1, 2, 3\right)$$

$$= \Phi\left(x_{1i}^T\beta\right) - \Phi\left(x_{0i}^T\beta\right)$$

$$= \Phi\left(\beta_0 + \beta_1 X_{01} + \beta_2 X_{02} + \beta_3 X_{02}^2 + \beta_4 X_{03} + \beta_5 + \beta_6 X_{03}\right) - \Phi\left(\beta_0 + \beta_1 X_{01} + \beta_2 X_{02} + \beta_3 X_{02}^2 + \beta_4 X_{03}\right)$$

$$= \Phi\left(\beta_0 + \beta_5 + \beta_1 X_{01} + \beta_2 X_{02} + \beta_3 X_{02}^2 + (\beta_4 + \beta_6) X_{03}\right) - \Phi\left(\beta_0 + \beta_1 X_{01} + \beta_2 X_{02} + \beta_3 X_{02}^2 + \beta_4 X_{03}\right)$$