Marginal Effects in Probit Models: Interpretation and Testing

This note introduces you to the two types of marginal effects in probit models: marginal index effects, and marginal probability effects. It demonstrates how to calculate these effects for both continuous and categorical explanatory variables.

1. Interpreting Probit Coefficients

A Generic Probit Model

- The conventional formulation of a binary dependent variable model assumes that an unobserved (or latent) dependent variable $Y_i^*$ is generated by a classical linear regression model of the form

$$Y_i^* = x_i^T \beta + u_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \cdots + \beta_k X_{ik} + u_i\quad(1)$$

where:

$Y_i^*$ = a continuous real-valued index variable for observation i that is unobservable, or latent;

$x_i^T = (1 \ X_{i1} \ X_{i2} \ \cdots \ X_{ik})$, a 1×K row vector of regressor values for observation i;

$\beta = (\beta_0 \ \beta_1 \ \beta_2 \ \cdots \ \beta_k)^T$, a K×1 column vector of regression coefficients;

$x_i^T \beta = a 1\times1$ scalar called the index function for observation i;

$u_i$ = an iid $N(0, \sigma^2)$ random error term for observation i.

- The observable outcomes of the binary choice problem are represented by a binary indicator variable $Y_i$ that is related to the unobserved dependent variable $Y_i^*$ as follows:

$$Y_i = 1 \text{ if } Y_i^* > 0\quad(2.1)$$

$$Y_i = 0 \text{ if } Y_i^* \leq 0\quad(2.2)$$
• **The random indicator variable** \( Y_i \) **represents the observed realizations of a binomial process with the following probabilities:**

\[
\begin{align*}
\Pr(Y_i = 1) &= \Pr(Y_i^* > 0) = \Pr(x_i^T \beta + u_i > 0) \\
\Pr(Y_i = 0) &= \Pr(Y_i^* \leq 0) = \Pr(x_i^T \beta + u_i \leq 0)
\end{align*}
\]

(3.1) (3.2)

• **Probit models** analytically represent the binomial probabilities (3.1) and (3.2) in terms of the standard normal c.d.f. \( \Phi(Z) \) as follows:

\[
\begin{align*}
\Pr(Y_i = 1) &= \Phi(x_i^T \beta) \\
\Pr(Y_i = 0) &= 1 - \Phi(x_i^T \beta)
\end{align*}
\]

(4.1) (4.2)

• **Interpretation of the probit coefficient vector** \( \beta \)

• Under the zero conditional mean error assumption, equation (1) implies that

\[
E(Y_i^* | x_i^T) = E(x_i^T \beta | x_i^T) + E(u_i | x_i^T) = x_i^T \beta \quad \text{since } E(u_i | x_i^T) = 0.
\]

(5)

• The **index function (or regression function)** \( x_i^T \beta \) is thus the conditional mean value of the latent random variable \( Y_i^* \) for given values of the regressors.

• The **slope coefficients** \( \beta_j \) (\( j = 1, \ldots, k \)): If all explanatory variables are **continuous** and enter the index function **linearly**, the partial derivatives of regression function (5) with respect to the individual regressors are the **slope coefficients** \( \beta_j \) (\( j = 1, \ldots, k \)):

\[
\frac{\partial E(Y_i^* | x_i^T)}{\partial x_{ij}} = \frac{\partial x_i^T \beta}{\partial x_{ij}} = \frac{\partial (\beta_0 + \beta_1 x_{i1} + \cdots + \beta_j x_{ij} + \cdots + \beta_k x_{ik})}{\partial x_{ij}} = \beta_j.
\]

• But if some of the explanatory variables are **binary** or enter the index function **nonlinearly**, the partial derivatives of regression function (5) are not so simply interpreted.
2. Two Types of Marginal Effects in Probit Models

For each explanatory variable, there are two types of marginal effects in binary dependent variables models.

**Marginal Index Effects**

Marginal index effects are the partial effects of each explanatory variable on the probit index function $x_i^T \beta$.

- **Case 1**: $X_j$ is a continuous explanatory variable

  The marginal index effect of a continuous explanatory variable $X_j$ is:

  $$\text{marginal index effect of variable } X_j = \frac{\partial E(Y_i^* | x_i^T)}{\partial X_{ij}} = \frac{\partial x_i^T \beta}{\partial X_{ij}}$$

- **Case 2**: $X_j$ is a binary explanatory variable (a dummy or indicator variable)

  The marginal index effect of a binary explanatory variable equals

  1. the value of the index function $x_i^T \beta$ when $X_{ij} = 1$ and the other regressors equal fixed values

  minus

  2. the value of the index function $x_i^T \beta$ when $X_{ij} = 0$ and the other regressors equal the same fixed values

**Formal Definition:** Define two different vectors of regressor values:

- $x_{i|1}^T = \text{any vector of regressor values with } X_{ij} = 1$;
- $x_{i|0}^T = \text{the same vector of regressor values but with } X_{ij} = 0$.

The marginal index effect of the binary (dummy) variable $X_j$ is:

$$\text{marginal index effect of } X_j = x_{i|1}^T \beta - x_{i|0}^T \beta$$
Limitation: Marginal index effects are difficult to interpret because it is difficult to interpret – and impossible to measure – the latent dependent variable $Y_i^*$.

Marginal Probability Effects

Marginal probability effects are the partial effects of each explanatory variable on the probability that the observed dependent variable $Y_i = 1$, where in probit models

$$\Pr(Y_i = 1) = \Phi(x_i^T \beta) = \text{standard normal c.d.f. evaluated at } x_i^T \beta.$$  

- Case 1: $X_j$ is a continuous explanatory variable

The marginal probability effect of variable $X_j$ is

$$\text{marginal probability effect of variable } X_j = \frac{\partial \Pr(Y_i = 1)}{\partial X_{ij}} = \frac{\partial \Phi(x_i^T \beta)}{\partial X_{ij}}$$

Using the chain rule of differentiation, we can obtain a general expression for the marginal probability effect of a continuous explanatory variable $X_j$:

$$\text{marginal probability effect of } X_j = \frac{\partial \Phi(x_i^T \beta)}{\partial X_{ij}} = \frac{\partial \Phi(x_i^T \beta)}{\partial (x_i^T \beta)} \cdot \frac{\partial (x_i^T \beta)}{\partial X_{ij}} = \phi(x_i^T \beta) \frac{\partial x_i^T \beta}{\partial X_{ij}}$$

where

$$\phi(x_i^T \beta) = \frac{\partial \Phi(x_i^T \beta)}{\partial (x_i^T \beta)} = \text{the value of the standard normal p.d.f. at } x_i^T \beta.$$  

$$\frac{\partial x_i^T \beta}{\partial X_{ij}} = \text{the marginal index effect of } X_j$$
Case 2: $X_j$ is a binary explanatory variable (a dummy or indicator variable)

The marginal probability effect of a binary explanatory variable equals

1. the value of $\Phi(x_i^T\beta)$ when $X_{ij} = 1$ and the other regressors equal fixed values minus
2. value of $\Phi(x_i^T\beta)$ when $X_{ij} = 0$ and the other regressors equal the same fixed values

Formal Definition: Define two different vectors of regressor values:

$x_{i_t} = \text{any vector of regressor values with } X_{ij} = 1$;
$x_{0_i} = \text{the same vector of regressor values but with } X_{ij} = 0$.

The marginal probability effect of the binary (dummy) variable $X_j$ is:

$$\text{marginal probability effect of } X_j = \Phi(x_{i_t}^T\beta) - \Phi(x_{0_i}^T\beta).$$

Relationship Between the Two Marginal Effects for Continuous Variables

- Compare the marginal index effect and marginal probability effect of a continuous explanatory variable $X_j$.

$$\text{marginal index effect of variable } X_j = \frac{\partial x_i^T\beta}{\partial X_{ij}}$$

$$\text{marginal probability effect of variable } X_j = \frac{\partial \Phi(x_i^T\beta)}{\partial X_{ij}} = \phi(x_i^T\beta) \frac{\partial x_i^T\beta}{\partial X_{ij}}$$

Relationship: For a continuous explanatory variable $X_j$, the marginal probability effect is proportional to the marginal index effect of $X_j$, where the factor of proportionality is the standard normal p.d.f. at $x_i^T\beta$:

$$\text{marginal probability effect of } X_j = \phi(x_i^T\beta) \times \text{marginal index effect of } X_j$$
3. Marginal Index and Probability Effects in Probit Models

A Simple Probit Model

\[ Y_i^* = x_i^T \beta + u_i = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i2}^2 + \beta_4 X_{i3} + \beta_5 D_i + \beta_6 D_i X_{i3} + u_i \]

where:

- \( X_{i1}, X_{i2} \) and \( X_{i3} \) are *continuous* explanatory variables
- \( D_i \) is a *binary* (or dummy) explanatory variable defined such that \( D_i = 1 \) if observation \( i \) exhibits some attribute, \( = 0 \) otherwise.

- **The index function** is:
  \[ x_i^T \beta = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i2}^2 + \beta_4 X_{i3} + \beta_5 D_i + \beta_6 D_i X_{i3} \]
  - \( X_{i1} \) enters the index function *linearly*.
  - \( X_{i2} \) enters the index function *nonlinearly*.
  - \( X_{i3} \) enters the index function *nonlinearly*, interacted with \( D_i \).
  - \( D_i \) enters the index function *nonlinearly*, interacted with \( X_{i3} \).

- **The observed binary dependent variable** \( Y_i \) is related to the unobserved dependent variable \( Y_i^* \) as follows:
  \[ Y_i = 1 \text{ if } Y_i^* > 0 \]
  \[ Y_i = 0 \text{ if } Y_i^* \leq 0 \]

- The **binomial probabilities** \( \Pr(Y_i = 1) \) and \( \Pr(Y_i = 0) \) are analytically represented in probit models in terms of the standard normal c.d.f. \( \Phi(Z) \):
  \[ \Pr(Y_i = 1) = \Pr(Y_i^* > 0) = \Phi(x_i^T \beta) = \Phi(\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i2}^2 + \beta_4 X_{i3} + \beta_5 D_i + \beta_6 D_i X_{i3}) \]
  \[ \Pr(Y_i = 0) = \Pr(Y_i^* \leq 0) = 1 - \Phi(x_i^T \beta) = 1 - \Phi(\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i2}^2 + \beta_4 X_{i3} + \beta_5 D_i + \beta_6 D_i X_{i3}) \]
Marginal Effects of $X_1$ = a continuous variable that enters linearly

\[ x_i^T \beta = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i3}^2 + \beta_4 X_{i3} + \beta_5 D_i + \beta_6 D_i X_{i3} \]

- **Marginal index effect of $X_1$**

  \[ \text{marginal index effect of } X_1 = \frac{\partial x_i^T \beta}{\partial X_{i1}} = \beta_1 \]

- **Marginal probability effect of $X_1$**

  \[ \text{marginal probability effect of } X_1 = \phi(x_i^T \beta) \frac{\partial x_i^T \beta}{\partial X_{i1}} = \phi(x_i^T \beta) \beta_1 \]

Marginal Effects of $X_2$ = a continuous variable that enters nonlinearly

- **Marginal index effect of $X_2$**

  \[ \text{marginal index effect of } X_2 = \frac{\partial x_i^T \beta}{\partial X_{i2}} = \beta_2 + 2 \beta_3 X_{i2} \]

- **Marginal probability effect of $X_2$**

  \[ \text{marginal probability effect of } X_2 = \phi(x_i^T \beta) \frac{\partial x_i^T \beta}{\partial X_{i2}} = \phi(x_i^T \beta) (\beta_2 + 2 \beta_3 X_{i2}) \]
Marginal Effects of $X_3$ = a continuous variable that enters non-linearly

$$x_i^T \beta = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i2}^2 + \beta_4 X_{i3} + \beta_5 D_i + \beta_6 D_i X_{i3}$$

- **Marginal index effect of $X_3$**

  marginal index effect of $X_3 = \frac{\partial x_i^T \beta}{\partial X_{i3}} = \beta_4 + \beta_6 D_i = \beta_4 + \beta_6$ when $D_i = 1$

  $$= \beta_4 \quad \text{when } D_i = 0$$

- **Marginal probability effect of $X_3$**

  marginal probability effect of $X_3 = \phi(x_i^T \beta) \frac{\partial x_i^T \beta}{\partial X_{i3}} = \phi(x_i^T \beta)(\beta_4 + \beta_6 D_i)$

  $$= \phi(x_i^T \beta)(\beta_4 + \beta_6) \quad \text{when } D_i = 1$$

  $$= \phi(x_i^T \beta) \beta_4 \quad \text{when } D_i = 0$$
Marginal Effects of $D$ = a binary variable that enters non-linearly

- **Marginal index effect of $D$**

  Define two vectors of regressor values that contain the same values of the other three explanatory variables, $X_{i1}$, $X_{i2}$ and $X_{i3}$:

  one with $D_i = 1$: $\mathbf{x}_i^T = \begin{pmatrix} 1 & X_{i1} & X_{i2} & X_{i3} & 1 & X_{i3} \end{pmatrix}$
  
  the other with $D_i = 0$: $\mathbf{x}_0^T = \begin{pmatrix} 1 & X_{i1} & X_{i2} & X_{i3} & 0 & 0 \end{pmatrix}$

  The corresponding **values of the index function** are:

  for $D_i = 1$: $\mathbf{x}_i^T \beta = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i3} + \beta_4 X_{i2} + \beta_5 + \beta_6 X_{i3}$
  
  $\quad = \beta_0 + \beta_5 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i2}^2 + (\beta_4 + \beta_6) X_{i3}$

  for $D_i = 0$: $\mathbf{x}_0^T \beta = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i2}^2 + \beta_4 X_{i3}$

  The **index function difference** = $\mathbf{x}_i^T \beta - \mathbf{x}_0^T \beta = \beta_5 + \beta_6 X_{i3}$

  The **marginal index effect of $D$** is:

  marginal index effect of $D = \mathbf{x}_i^T \beta - \mathbf{x}_0^T \beta = \beta_5 + \beta_6 X_{i3}$

- **Marginal probability effect of $D$**

  marginal probability effect of $D = \Phi(\mathbf{x}_i^T \beta) - \Phi(\mathbf{x}_0^T \beta)$

  $\quad = \Phi\left(\beta_0 + \beta_5 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i2}^2 + (\beta_4 + \beta_6) X_{i3}\right)$
  
  $\quad \quad - \Phi\left(\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \beta_3 X_{i2}^2 + \beta_4 X_{i3}\right)$