Solution to Exercise 9.22

⋆9.22 Describe the two procedures by which the parameters μ and σ^2 of the lognormal distribution can be estimated by the method of simulated moments, matching the first and second moments of the lognormal variable itself, and the first moment of its log. The first procedure should use optimal instruments and be just identified; the second should use the simple instruments of (9.108) and be overidentified.

In terms of the notation used in the text, we wish to match the moments which are the expectations of the z_t, y_t, and y_t^2. The elementary zero functions for MSM estimation are therefore

$$\begin{aligned}
f^*_{t1}(z_t, \mu, \sigma) &= z_t - \frac{1}{S}\sum_{s=1}^{S} m^*_1(u^*_{ts}, \mu, \sigma),\\
f^*_{t2}(y_t, \mu, \sigma) &= y_t - \frac{1}{S}\sum_{s=1}^{S} m^*_2(u^*_{ts}, \mu, \sigma), \text{ and}\\
f^*_{t3}(y_t^2, \mu, \sigma) &= y_t^2 - \frac{1}{S}\sum_{s=1}^{S} m^*_3(u^*_{ts}, \mu, \sigma),
\end{aligned}$$

where

$$\begin{aligned}
m^*_1(u^*, \mu, \sigma) &\equiv \mu + \sigma u^*,\\
m^*_2(u^*, \mu, \sigma) &\equiv \exp(\mu + \sigma u^*), \text{ and}\\
m^*_3(u^*, \mu, \sigma) &\equiv \exp(2\mu + 2\sigma u^*),
\end{aligned}$$

and the u^*_{ts} are IID standard normal. We write $\boldsymbol{f}_i(\mu, \sigma^2)$, $i = 1, 2, 3$, for the n−vectors with the f^*_{ti} as typical elements, and $\boldsymbol{f}(\cdot) = [\boldsymbol{f}_1(\cdot) \;\vdots\; \boldsymbol{f}_2(\cdot) \;\vdots\; \boldsymbol{f}_3(\cdot)]$.

The Jacobian matrix of $\boldsymbol{f}(\mu, \sigma^2)$ with respect to μ and σ is a $3n \times 2$ matrix that we write as $\boldsymbol{F}^*(\boldsymbol{U}^*, \mu, \sigma)$, $\boldsymbol{U}^*$ being the collection of all the u^*_{ts}, $t = 1, \ldots, n$, $s = 1, \ldots, S$. The typical elements of $\boldsymbol{F}^*$ are given in the matrix

$$-\frac{1}{S}\sum_{s=1}^{S}\begin{bmatrix} 1 & u^*_{ts} \\ e^{\mu+\sigma u^*_{ts}} & u^*_{ts} e^{\mu+\sigma u^*_{ts}} \\ 2e^{2\mu+2\sigma u^*_{ts}} & 2u^*_{ts} e^{2\mu+2\sigma u^*_{ts}} \end{bmatrix}. \tag{S9.48}$$

As in (9.105), the $3n \times 3n$ matrix $\boldsymbol{\Omega}$ of variances and covariances of the elementary zero functions takes the form

$$\boldsymbol{\Omega} = \begin{bmatrix} \sigma_{11}\mathbf{I} & \sigma_{12}\mathbf{I} & \sigma_{13}\mathbf{I} \\ \sigma_{21}\mathbf{I} & \sigma_{22}\mathbf{I} & \sigma_{23}\mathbf{I} \\ \sigma_{31}\mathbf{I} & \sigma_{32}\mathbf{I} & \sigma_{33}\mathbf{I} \end{bmatrix},$$

where $\sigma_{11} = \text{Var}(z_t)$, $\sigma_{12} = \sigma_{21} = \text{Cov}(z_t, y_t)$, $\sigma_{13} = \sigma_{31} = \text{Cov}(z_t, y_t^2)$, $\sigma_{22} = \text{Var}(y_t)$, $\sigma_{23} = \sigma_{32} = \text{Cov}(y_t, y_t^2)$, and $\sigma_{33} = \text{Var}(y_t^2)$. The σ_{ij} can be estimated consistently as the sample variances and covariances of the z_t, y_t, and y_t^2.

The optimal instruments for this problem are given by the two columns of $\boldsymbol{\Omega}^{-1}\text{E}\big(\boldsymbol{F}(\mu_0, \sigma_0)\big)$, where μ_0 and σ_0 are the unknown true values. Note that the expectation here is with respect to simulation randomness only, since the elements of the matrix (S9.48) depend neither on the z_t nor on the y_t. Let $\boldsymbol{\Sigma}$ be the 3×3 matrix with typical element σ_{ij}, $i, j = 1, 2, 3$. Denote by σ^{ij} the typical element of $\boldsymbol{\Sigma}^{-1}$. Then the optimal instrument matrix is, up to a sign change,

$$\begin{bmatrix} \sigma^{11}\mathbf{I} & \sigma^{12}\mathbf{I} & \sigma^{13}\mathbf{I} \\ \sigma^{21}\mathbf{I} & \sigma^{22}\mathbf{I} & \sigma^{23}\mathbf{I} \\ \sigma^{31}\mathbf{I} & \sigma^{32}\mathbf{I} & \sigma^{33}\mathbf{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{\iota} & f_{12}(\mu, \sigma)\boldsymbol{\iota} \\ f_{21}(\mu, \sigma)\boldsymbol{\iota} & f_{22}(\mu, \sigma)\boldsymbol{\iota} \\ f_{31}(\mu, \sigma)\boldsymbol{\iota} & f_{32}(\mu, \sigma)\boldsymbol{\iota} \end{bmatrix},$$

evaluated at $\mu = \mu_0$ and $\sigma = \sigma_0$. Here the fact that the u^*_{ts} are IID implies that each block in $\text{E}(\boldsymbol{F}(\cdot))$ is proportional to $\boldsymbol{\iota}$, and the f_{ij} are the expectations of the elements of (S9.48). Even if we assume no knowledge of the analytic form of the moments, it is reasonable to note that $f_{12}(\mu, \sigma) \equiv \text{E}(u^*_{ts}) = 0$.

In practice, we need estimates of the σ_{ij}, or, equivalently, of the σ^{ij}, estimates of the f_{ij}, for $i > 1$, and estimates of μ and σ. The last of these merely have to be consistent, and they can be obtained in a variety of ways, for instance by solving the two equations (9.107), or by minimizing a criterion function based on all three moments, but with an identity weighting matrix. We have already mentioned that the σ_{ij} can be estimated as sample variances and covariances. They could also be estimated by simulation, using the preliminary estimates $\hat{\mu}$ and $\hat{\sigma}$. For f_{21}, the easiest estimator is

$$\hat{f}_{21}(\mu, \sigma) = \frac{1}{nS} \sum_{t=1}^{n} \sum_{s=1}^{S} \exp(\mu + \sigma u^*_{ts}),$$

with similar expressions for the other f_{ij}, all to be evaluated at $(\hat{\mu}, \hat{\sigma})$. The estimates of the optimal instruments are therefore given by the $3n \times 2$ matrix

$$\hat{\boldsymbol{W}} \equiv \begin{bmatrix} (\hat{\sigma}^{11} + \hat{\sigma}^{12}\hat{f}_{21} + \hat{\sigma}^{13}\hat{f}_{31})\boldsymbol{\iota} & (\hat{\sigma}^{12}\hat{f}_{22} + \hat{\sigma}^{13}\hat{f}_{32})\boldsymbol{\iota} \\ (\hat{\sigma}^{21} + \hat{\sigma}^{22}\hat{f}_{21} + \hat{\sigma}^{23}\hat{f}_{31})\boldsymbol{\iota} & (\hat{\sigma}^{22}\hat{f}_{22} + \hat{\sigma}^{23}\hat{f}_{32})\boldsymbol{\iota} \\ (\hat{\sigma}^{31} + \hat{\sigma}^{32}\hat{f}_{21} + \hat{\sigma}^{33}\hat{f}_{31})\boldsymbol{\iota} & (\hat{\sigma}^{32}\hat{f}_{22} + \hat{\sigma}^{33}\hat{f}_{32})\boldsymbol{\iota} \end{bmatrix}. \tag{S9.49}$$

If we make the definitions

$$\bar{m}_i(\mu, \sigma) = \frac{1}{nS} \sum_{t=1}^{n} \sum_{s=1}^{S} m^*_i(u^*_{ts}, \mu, \sigma),$$

for $i = 1, 2, 3$, then the estimating equations, to be solved for $\hat{\mu}$ and $\hat{\sigma}$, are

$$\begin{aligned}&\big(\hat{\sigma}^{11} + \hat{\sigma}^{12}\hat{f}_{21} + \hat{\sigma}^{13}\hat{f}_{31}\big)\big(\bar{z} - \bar{m}_1(\mu, \sigma)\big) + \\ &\quad \big(\hat{\sigma}^{21} + \hat{\sigma}^{22}\hat{f}_{21} + \hat{\sigma}^{23}\hat{f}_{31}\big)\big(\bar{y} - \bar{m}_2(\mu, \sigma)\big) + \\ &\qquad \big(\hat{\sigma}^{31} + \hat{\sigma}^{32}\hat{f}_{21} + \hat{\sigma}^{33}\hat{f}_{31}\big)\big(\bar{y^2} - \bar{m}_3(\mu, \sigma)\big) = 0,\end{aligned}$$

and

$$\begin{aligned}&\big(\hat{\sigma}^{12}\hat{f}_{22} + \hat{\sigma}^{13}\hat{f}_{32}\big)\big(\bar{z} - \bar{m}_1(\mu, \sigma)\big) + \\ &\quad \big(\hat{\sigma}^{22}\hat{f}_{22} + \hat{\sigma}^{23}\hat{f}_{32}\big)\big(\bar{y} - \bar{m}_2(\mu, \sigma)\big) + \\ &\qquad \big(\hat{\sigma}^{32}\hat{f}_{22} + \hat{\sigma}^{33}\hat{f}_{32}\big)\big(\bar{y^2} - \bar{m}_3(\mu, \sigma)\big) = 0.\end{aligned}$$

Here $\bar{z}$, $\bar{y}$, and $\bar{y^2}$ are the sample means of the z_t, y_t, and y_t^2 respectively. In principle, it is possible to write these equations with the optimal instruments explicitly as functions of μ and σ, and then solve the estimating equations with unknown μ and σ in both the instruments and the zero functions. This would be rather hard to write out and program, and there could be numerical problems in applying Newton's Method to the resulting equations. If preliminary estimates of μ and σ are used, then the estimation procedure may be iterated.

The other method of estimation makes use of the instruments (9.108), which, in combination with the vectors $\boldsymbol{f}_i(\cdot)$ of elementary zero functions, give rise to the sample moments

$$\bar{z} - \bar{m}_1(\mu, \sigma), \;\; \bar{y} - \bar{m}_2(\mu, \sigma) \;\text{ and }\; \bar{y^2} - \bar{m}_3(\mu, \sigma).$$

The covariance matrix of these sample moments is proportional to the covariance matrix of z_t, y_t, and y_t^2, which is just the matrix we called $\boldsymbol{\Sigma}$. A criterion function can thus be formed using the three sample moments and $\hat{\boldsymbol{\Sigma}}^{-1}$. Minimizing it with respect to μ and σ yields the overidentified MSM estimates.

Both methods yield asymptotically equivalent estimators. The asymptotic covariance matrix is given by (9.116), with $\boldsymbol{W}$ given by (9.108). In this expression, $\hat{\boldsymbol{F}}$ can be replaced by an estimate of $\mathrm{E}\big(\boldsymbol{F}(\hat{\mu}, \hat{\sigma})\big)$, which is the matrix

$$\begin{bmatrix} \boldsymbol{\iota} & \mathbf{0} \\ \hat{f}_{21}\boldsymbol{\iota} & \hat{f}_{22}\boldsymbol{\iota} \\ \hat{f}_{31}\boldsymbol{\iota} & \hat{f}_{32}\boldsymbol{\iota} \end{bmatrix}.$$

Alternatively, (9.116) can be interpreted as applying to the just-identified method, with instruments (S9.49). This has the advantage of eliminating the apparent sandwich.