Solution to Exercise 9.10

⋆9.10 Give the explicit form of the $n \times n$ matrix $\boldsymbol{U}(j)$ for which $\hat{\boldsymbol{\Gamma}}(j)$, defined in (9.36), takes the form $n^{-1}\boldsymbol{W}^\top\boldsymbol{U}(j)\boldsymbol{W}$.

A typical element of $n^{-1}\boldsymbol{W}^\top\boldsymbol{U}(j)\boldsymbol{W}$ is

$$\frac{1}{n}\sum_{t=1}^{n}\sum_{s=1}^{n} W_{ti}W_{tk}U_{ts}(j) \tag{S9.26}$$

If we make the definition

$$U_{t,t-j}(j) = \hat{u}_t\hat{u}_{t-j}, \quad \text{for } \ t = j+1, \ldots, n, \ \ U_{ts}(j) = 0 \ \text{ otherwise,} \tag{S9.27}$$

then (S9.26) becomes

$$\frac{1}{n}\sum_{t=j+1}^{n} W_{ti}W_{t-j,k}\,\hat{u}_t\hat{u}_{t-j},$$

which is a typical element of $\hat{\boldsymbol{\Gamma}}(j)$ as defined in (9.36). Thus (S9.27) provides the form of the matrix $\boldsymbol{U}(j)$ asked for in the question.