Solution to Exercise 8.5

⋆8.5 Under the usual assumptions of this chapter, including (8.16), show that the plim of

$$\frac{1}{n}Q(\beta_0,\boldsymbol{y}) = \frac{1}{n}(\boldsymbol{y}-\boldsymbol{X}\beta_0)^\top\boldsymbol{P_W}(\boldsymbol{y}-\boldsymbol{X}\beta_0)$$

is zero if $\boldsymbol{y}=\boldsymbol{X}\beta_0+\boldsymbol{u}$. Under the same assumptions, along with the asymptotic identification condition that $\boldsymbol{S}_{\boldsymbol{X}^\top\boldsymbol{W}}(\boldsymbol{S}_{\boldsymbol{W}^\top\boldsymbol{W}})^{-1}\boldsymbol{S}_{\boldsymbol{W}^\top\boldsymbol{X}}$ has full rank, show further that $\operatorname{plim} n^{-1}Q(\boldsymbol{\beta},\boldsymbol{y})$ is strictly positive for $\boldsymbol{\beta}\neq\boldsymbol{\beta}_0$.

When $\boldsymbol{y}=\boldsymbol{X\beta}_0+\boldsymbol{u}$, the probability limit of $n^{-1}Q(\boldsymbol{\beta},\boldsymbol{y})$ is

$$\begin{aligned}\operatorname*{plim}_{n\to\infty}\frac{1}{n}Q(\boldsymbol{\beta}_0,\boldsymbol{y}) &= \operatorname*{plim}_{n\to\infty}\frac{1}{n}\boldsymbol{u}^\top\boldsymbol{P_W}\boldsymbol{u} \qquad \text{(S8.02)}\\ &= \Big(\operatorname*{plim}_{n\to\infty}\frac{1}{n}\boldsymbol{u}^\top\boldsymbol{W}\Big)\Big(\operatorname*{plim}_{n\to\infty}\frac{1}{n}\boldsymbol{W}^\top\boldsymbol{W}\Big)^{-1}\Big(\operatorname*{plim}_{n\to\infty}\frac{1}{n}\boldsymbol{W}^\top\boldsymbol{u}\Big).\end{aligned}$$

A standard assumption for the IV estimator to be consistent is (8.16), according to which $\operatorname{plim} n^{-1}\boldsymbol{W}^\top\boldsymbol{u}=\boldsymbol{0}$. It is also assumed that $\operatorname{plim} n^{-1}\boldsymbol{W}^\top\boldsymbol{W} = \boldsymbol{S}_{\boldsymbol{W}^\top\boldsymbol{W}}$, which is a nonsingular matrix. Under these assumptions, the second line of (S8.02) implies that

$$\operatorname*{plim}_{n\to\infty}\frac{1}{n}Q(\boldsymbol{\beta}_0,\boldsymbol{y}) = \boldsymbol{0}^\top\boldsymbol{S}^{-1}_{\boldsymbol{W}^\top\boldsymbol{W}}\boldsymbol{0}=0.$$

This answers the first part of the question.

When $\boldsymbol{y}=\boldsymbol{X\beta}_0+\boldsymbol{u}$, we see that

$$\boldsymbol{y}-\boldsymbol{X\beta} = \boldsymbol{X\beta}_0+\boldsymbol{u}-\boldsymbol{X\beta} = \boldsymbol{u}+\boldsymbol{X}(\boldsymbol{\beta}_0-\boldsymbol{\beta}).$$

Therefore, the criterion function can be written as

$$Q(\boldsymbol{\beta}_0,\boldsymbol{y}) = \boldsymbol{u}^\top\boldsymbol{P_W}\boldsymbol{u} + 2\boldsymbol{u}^\top\boldsymbol{P_W X}(\boldsymbol{\beta}_0-\boldsymbol{\beta}) + (\boldsymbol{\beta}_0-\boldsymbol{\beta})^\top\boldsymbol{X}^\top\boldsymbol{P_W X}(\boldsymbol{\beta}_0-\boldsymbol{\beta}).$$

We have just seen that the plim of n^{-1} times the first term here is zero. By a similar argument, so is the plim of n^{-1} times the second term. Therefore,

$$\operatorname*{plim}_{n\to\infty}\frac{1}{n}Q(\boldsymbol{\beta}_0,\boldsymbol{y}) = \operatorname*{plim}_{n\to\infty}\frac{1}{n}(\boldsymbol{\beta}_0-\boldsymbol{\beta})^\top\boldsymbol{X}^\top\boldsymbol{P_W X}(\boldsymbol{\beta}_0-\boldsymbol{\beta}).$$

This can be written as the product of three probability limits:

$$\Big(\operatorname*{plim}_{n\to\infty}\frac{1}{n}(\boldsymbol{\beta}_0-\boldsymbol{\beta})^\top\boldsymbol{X}^\top\boldsymbol{W}\Big)\Big(\operatorname*{plim}_{n\to\infty}\frac{1}{n}\boldsymbol{W}^\top\boldsymbol{W}\Big)^{-1}\Big(\operatorname*{plim}_{n\to\infty}\frac{1}{n}\boldsymbol{W}^\top\boldsymbol{X}(\boldsymbol{\beta}_0-\boldsymbol{\beta})\Big),$$

Using the definitions of $\boldsymbol{S}_{\boldsymbol{X}^\top\boldsymbol{W}}$, $\boldsymbol{S}_{\boldsymbol{W}^\top\boldsymbol{X}}$, and $\boldsymbol{S}_{\boldsymbol{W}^\top\boldsymbol{W}}$, we conclude that

$$\operatorname*{plim}_{n\to\infty}\frac{1}{n}Q(\boldsymbol{\beta}_0,\boldsymbol{y}) = (\boldsymbol{\beta}_0-\boldsymbol{\beta})^\top\boldsymbol{S}_{\boldsymbol{X}^\top\boldsymbol{W}}(\boldsymbol{S}_{\boldsymbol{W}^\top\boldsymbol{W}})^{-1}\boldsymbol{S}_{\boldsymbol{W}^\top\boldsymbol{X}}(\boldsymbol{\beta}_0-\boldsymbol{\beta}).$$

If the matrix $\boldsymbol{S}_{\boldsymbol{X}^\top\boldsymbol{W}}(\boldsymbol{S}_{\boldsymbol{W}^\top\boldsymbol{W}})^{-1}\boldsymbol{S}_{\boldsymbol{W}^\top\boldsymbol{X}}$ has full rank, as we have assumed, this is a positive definite quadratic form. Therefore, the probability limit of $n^{-1}Q(\boldsymbol{\beta}_0,\boldsymbol{y})$ must be positive whenever $\boldsymbol{\beta}\neq\boldsymbol{\beta}_0$.