Solution to Exercise 7.26

⋆7.26 Suppose that, in the error-components model (7.100), none of the columns of $\boldsymbol{X}$ displays any within-group variation. Recall that, for this model, the data are balanced, with m groups and T observations per group. Show that the OLS and GLS estimators are identical in this special case. Then write down the true covariance matrix of both these estimators. How is this covariance matrix related to the usual one for OLS that would be computed by a regression package under classical assumptions? What happens to this relationship as T and ρ, the correlation of the error terms within groups, change?

In Exercise 7.25, we saw that

$$\boldsymbol{\Sigma}^{-1} = \frac{1}{\sigma_\varepsilon^2}\big(\mathbf{I}_T - (2\lambda - \lambda^2)\boldsymbol{P_\iota}\big),$$

where $\boldsymbol{\Sigma}$ is the covariance matrix for each group of T observations. After a little algebra, this can be rewritten as

$$\boldsymbol{\Sigma}^{-1} = \frac{1}{\sigma_\varepsilon^2}\Big(\mathbf{I}_T - \frac{T\sigma_v^2}{T\sigma_v^2 + \sigma_\varepsilon^2}\boldsymbol{P_\iota}\Big).$$

Because there is no within-group variation, each of the matrices $\boldsymbol{X}_i$ consists of the same row repeated T times. Therefore, $\boldsymbol{P_\iota X}_i = \boldsymbol{X}_i$ for all i. From this it follows that

$$\begin{aligned}\boldsymbol{\Sigma}^{-1}\boldsymbol{X}_i &= \frac{1}{\sigma_\varepsilon^2}\boldsymbol{X}_i - \frac{1}{\sigma_\varepsilon^2}\frac{T\sigma_v^2}{T\sigma_v^2 + \sigma_\varepsilon^2}\boldsymbol{X}_i \\ &= \frac{1}{T\sigma_v^2 + \sigma_\varepsilon^2}\boldsymbol{X}_i.\end{aligned} \tag{S7.36}$$

The OLS estimator of $\boldsymbol{\beta}$ is just

$$\hat{\boldsymbol{\beta}}_{\text{OLS}} = (\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}^\top\boldsymbol{y} = \Big(\sum_{i=1}^m \boldsymbol{X}_i^\top\boldsymbol{X}_i\Big)^{-1}\sum_{i=1}^m \boldsymbol{X}_i^\top\boldsymbol{y}_i.$$

From (S7.36), the GLS estimator is

$$\begin{aligned}\hat{\boldsymbol{\beta}}_{\text{GLS}} &= (\boldsymbol{X}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{X})^{-1}\boldsymbol{X}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{y} \\ &= \Big(\frac{1}{T\sigma_v^2 + \sigma_\varepsilon^2}\sum_{i=1}^m \boldsymbol{X}_i^\top\boldsymbol{X}_i\Big)^{-1}\frac{1}{T\sigma_v^2 + \sigma_\varepsilon^2}\sum_{i=1}^m \boldsymbol{X}_i^\top\boldsymbol{y}_i,\end{aligned}$$

which is equal to $\hat{\boldsymbol{\beta}}_{\text{OLS}}$ since the scalar factors multiplying the sums cancel. Notice that the between-groups estimator (7.90) is also equal to the OLS and GLS estimators in this special case, because $\boldsymbol{P_D X} = \boldsymbol{X}$.

In this case, the true covariance matrix of the GLS estimator, which must also be the true covariance matrix of the OLS and between-groups estimators (when the last of these is run as a regression with m observations), is

$$(\boldsymbol{X}^\top\boldsymbol{\Omega}^{-1}\boldsymbol{X})^{-1} = (T\sigma_v^2 + \sigma_\varepsilon^2)(\boldsymbol{X}^\top\boldsymbol{X})^{-1}. \tag{S7.37}$$

In contrast, the usual OLS covariance matrix that would be printed by a regression package is

$$s^2(\boldsymbol{X}^\top\boldsymbol{X})^{-1},$$

where s^2 is an estimate of $\sigma_u^2 \equiv \sigma_v^2 + \sigma_\varepsilon^2$. This estimate will differ somewhat from σ_u^2 because of estimation error. However, at least in large samples, we can expect that, on average, s^2 and σ_u^2 will be similar. Thus the ratio of the true covariance matrix (S7.37) to the usual OLS covariance matrix is approximately

$$\frac{T\sigma_v^2 + \sigma_\varepsilon^2}{\sigma_v^2 + \sigma_\varepsilon^2} = \rho(T-1) + 1,$$

where $\rho \equiv \sigma_v^2/(\sigma_v^2 + \sigma_\varepsilon^2)$ is the correlation of error terms within groups. We can see that the usual OLS covariance matrix is correct only if $\rho = 0$ or $T = 1$, so that there are no group effects. As either ρ or T increases, the ratio of the true covariance matrix to the conventional one increases. If T is large, this ratio can be large even when ρ is small. For example, if $\rho = 0.05$ and $T = 100$, the ratio is 5.95. This implies that OLS standard errors are too small by a factor of 2.44.