Solution to Exercise 7.25

⋆7.25 Show that, for $\boldsymbol{\Sigma}$ defined in (7.88),

$$\boldsymbol{\Sigma}^{-1/2} = \frac{1}{\sigma_\varepsilon}(\mathbf{I}_T - \lambda \boldsymbol{P}_{\boldsymbol{\iota}}), \tag{S7.35}$$

where $\boldsymbol{P}_{\boldsymbol{\iota}} \equiv \boldsymbol{\iota}(\boldsymbol{\iota}^\top\boldsymbol{\iota})^{-1}\boldsymbol{\iota}^\top = (1/T)\boldsymbol{\iota}\boldsymbol{\iota}^\top$, and

$$\lambda = 1 - \left(\frac{T\sigma_v^2}{\sigma_\varepsilon^2} + 1\right)^{-1/2}.$$

Then use this result to show that the GLS estimates of $\boldsymbol{\beta}$ may be obtained by running regression (7.92). What is the covariance matrix of the GLS estimator?

From (7.88), we have

$$\boldsymbol{\Sigma} = \sigma_\varepsilon^2\big(\mathbf{I}_T + (\sigma_v^2/\sigma_\varepsilon^2)\boldsymbol{\iota}\boldsymbol{\iota}^\top\big) = \sigma_\varepsilon^2\big(\mathbf{I}_T + (T\sigma_v^2/\sigma_\varepsilon^2)\boldsymbol{P}_{\boldsymbol{\iota}}\big).$$

Let us denote the $T \times T$ matrix in parentheses in the rightmost expression above by $\boldsymbol{V}$. Suppose that $\boldsymbol{V}^{-1/2}$ can be written in the form $\mathbf{I} - \lambda\boldsymbol{P}_{\boldsymbol{\iota}}$, for some suitable λ. Then $\boldsymbol{V}^{-1}$ is equal to

$$(\mathbf{I} - \lambda\boldsymbol{P}_{\boldsymbol{\iota}})^2 = \mathbf{I} - (2\lambda - \lambda^2)\boldsymbol{P}_{\boldsymbol{\iota}},$$

since $\boldsymbol{P}_{\boldsymbol{\iota}}$ is idempotent. It follows that

$$\mathbf{I} = \boldsymbol{V}^{-1}\boldsymbol{V} = \big(\mathbf{I} - (2\lambda - \lambda^2)\boldsymbol{P}_{\boldsymbol{\iota}}\big)\big(\mathbf{I} + (T\sigma_v^2/\sigma_\varepsilon^2)\boldsymbol{P}_{\boldsymbol{\iota}}\big).$$

We require that the coefficient of $\boldsymbol{P}_{\boldsymbol{\iota}}$ in the expression above should vanish. Let $\tau \equiv T\sigma_v^2/\sigma_\varepsilon^2$. Then the requirement can be written as $\lambda^2 - 2\lambda + \tau/(1+\tau) = 0$. The solutions to this quadratic equation for λ are

$$\lambda = 1 \pm \left(1 - \frac{\tau}{1+\tau}\right)^{1/2} = 1 \pm \frac{1}{(T\sigma_v^2/\sigma_\varepsilon^2 + 1)^{1/2}}.$$

The solution with the minus sign can easily be rewritten as the λ in the statement of the question.

Given the result (S7.35), it follows that the regression (7.92) has exactly the form of the transformed regression (7.03) except for the irrelevant factor of σ_ε. Thus running (7.92) gives the GLS estimator.

Of course, we cannot ignore the factor of $1/\sigma_\varepsilon$ when we compute the GLS covariance matrix estimator. By the result (7.05), the estimator is

$$\mathrm{Var}(\hat{\boldsymbol{\beta}}_{\mathrm{GLS}}) = \sigma_\varepsilon^2\big(\boldsymbol{X}^\top(\mathbf{I} - \lambda\boldsymbol{P}_{\boldsymbol{D}})^2\boldsymbol{X}\big)^{-1}.$$

In the case of feasible GLS estimation, we need to replace both λ and σ_ε^2 by consistent estimators.