Solution to Exercise 7.13

⋆7.13 Consider testing for first-order serial correlation of the error terms in the regression model

$$\boldsymbol{y} = \beta \boldsymbol{y}_1 + \boldsymbol{u}, \quad |\beta| < 1, \tag{7.97}$$

where $\boldsymbol{y}_1$ is the vector with typical element y_{t-1}, by use of the statistics t_{GNR} and t_{SR} defined in (7.51) and (7.52), respectively. Show first that the vector denoted as $\boldsymbol{M_X}\tilde{\boldsymbol{u}}_1$ in (7.51) and (7.52) is equal to $-\tilde{\beta}\boldsymbol{M_X y}_2$, where $\boldsymbol{y}_2$ is the vector with typical element y_{t-2}, and $\tilde{\beta}$ is the OLS estimate of β from (7.97). Then show that, as $n \to \infty$, t_{GNR} tends to the random variable $\tau \equiv \sigma_u^{-2} \operatorname{plim} n^{-1/2}(\beta \boldsymbol{y}_1 - \boldsymbol{y}_2)^{\top}\boldsymbol{u}$, whereas t_{SR} tends to the same random variable times β. Show finally that t_{GNR}, but not t_{SR}, provides an asymptotically correct test, by showing that the random variable τ is asymptotically distributed as $\text{N}(0, 1)$.

For the model (7.97), $\tilde{\boldsymbol{u}}_1 = \boldsymbol{y}_1 - \tilde{\beta}\boldsymbol{y}_2$. Since the projection matrix $\boldsymbol{M_X}$ is equal to $\mathbf{I} - \boldsymbol{y}_1(\boldsymbol{y}_1^{\top}\boldsymbol{y}_1)^{-1}\boldsymbol{y}_1^{\top}$ in this case, it annihilates $\boldsymbol{y}_1$. Therefore,

$$\boldsymbol{M_X}\tilde{\boldsymbol{u}}_1 = \boldsymbol{M_X y}_1 - \boldsymbol{M_X}\tilde{\beta}\boldsymbol{y}_2 = -\tilde{\beta}\boldsymbol{M_X y}_2, \tag{S7.16}$$

which is the first result that was to be proved.

The numerator of both statistics is

$$n^{-1/2}\tilde{\boldsymbol{u}}^{\top}\boldsymbol{M_X}\tilde{\boldsymbol{u}}_1 = -n^{-1/2}\tilde{\beta}\boldsymbol{u}^{\top}\boldsymbol{M_X y}_2, \tag{S7.17}$$

where we have used (S7.16) and the standard result that $\tilde{\boldsymbol{u}} = \boldsymbol{M_X u}$. If we substitute $\mathbf{I} - \boldsymbol{y}_1(\boldsymbol{y}_1^{\top}\boldsymbol{y}_1)^{-1}\boldsymbol{y}_1$ for $\boldsymbol{M_X}$ in (S7.17), the right-hand side becomes

$$-n^{-1/2}\tilde{\beta}\big(\boldsymbol{u}^{\top}\boldsymbol{y}_2 - \boldsymbol{u}^{\top}\boldsymbol{y}_1(\boldsymbol{y}_1^{\top}\boldsymbol{y}_1)^{-1}\boldsymbol{y}_1^{\top}\boldsymbol{y}_2\big).$$

Because $\tilde{\beta}$ and $(\boldsymbol{y}_1^{\top}\boldsymbol{y}_1)^{-1}\boldsymbol{y}_1^{\top}\boldsymbol{y}_2$ both tend to β as $n \to \infty$, the probability limit of this expression is the random variable

$$\operatorname*{plim}_{n\to\infty} n^{-1/2}\beta\boldsymbol{u}^{\top}(\beta\boldsymbol{y}_1 - \boldsymbol{y}_2). \tag{S7.18}$$

The denominator of t_{GNR} is

$$s(n^{-1}\tilde{\boldsymbol{u}}_1^{\top}\boldsymbol{M_X}\tilde{\boldsymbol{u}}_1)^{1/2}.$$

The first factor evidently tends to σ_u under the null hypothesis, since $\tilde{\boldsymbol{u}}_1$ has no explanatory power for $\tilde{\boldsymbol{u}}$, asymptotically. The second factor is the square root of

$$n^{-1}\tilde{\boldsymbol{u}}_1^{\top}\boldsymbol{M_X}\tilde{\boldsymbol{u}}_1 = n^{-1}\tilde{\beta}^2\boldsymbol{y}_2^{\top}\boldsymbol{M_X y}_2,$$

where the equality follows from (S7.16). By essentially the same argument as the one that led to (S7.18), this is asymptotically equal to

$$n^{-1}\beta^2\boldsymbol{y}_2^{\top}\boldsymbol{y}_2 - n^{-1}\beta^4\boldsymbol{y}_1^{\top}\boldsymbol{y}_1. \tag{S7.19}$$

Using the fact that y_t has variance $\sigma_u^2/(1-\beta^2)$, it is easy to see that the limit in probability of both $n^{-1}\boldsymbol{y}_1^{\top}\boldsymbol{y}_1$ and $n^{-1}\boldsymbol{y}_2^{\top}\boldsymbol{y}_2$ is $\sigma_u^2/(1-\beta^2)$. Therefore, expression (S7.19) is asymptotically equal to

$$\frac{\sigma_u^2}{1-\beta^2}(\beta^2-\beta^4) = \frac{\sigma_u^2\beta^2(1-\beta^2)}{1-\beta^2} = \beta^2\sigma_u^2. \tag{S7.20}$$

Thus the denominator of $t_{\rm GNR}$ is asymptotically equal to σ_u times the square root of (S7.20), or $\beta\sigma_u^2$. Transposing (S7.18), which is of course a scalar, and dividing it by this denominator yields the desired result that

$$\operatorname*{plim}_{n\to\infty} t_{\rm GNR} \equiv \tau = \sigma_u^{-2}\operatorname{plim} n^{-1/2}(\beta\boldsymbol{y}_1 - \boldsymbol{y}_2)^{\top}\boldsymbol{u}. \tag{S7.21}$$

The analysis for $t_{\rm SR}$ is similar to, but simpler than, the one for $t_{\rm GNR}$. Both statistics have the same numerator, which is asymptotically equal to (S7.18). The first factor in the denominator of $t_{\rm SR}$ is $\acute{s}$, which evidently tends to σ_u under the null hypothesis, and the second factor is $(n^{-1}\tilde{\boldsymbol{u}}_1^{\top}\tilde{\boldsymbol{u}}_1)^{1/2}$. Since n^{-1} times the sum of squares of the residuals lagged once must be asymptotically the same as n^{-1} times the sum of squared residuals, this second factor clearly has a probability limit of

$$\operatorname*{plim}_{n\to\infty}(n^{-1}\boldsymbol{u}^{\top}\boldsymbol{M_X}\boldsymbol{u})^{1/2} = \sigma_u$$

under the null hypothesis. Therefore, the denominator of $t_{\rm SR}$ is asymptotically equal to $1/\beta$ times the denominator of $t_{\rm GNR}$, and we find that

$$\operatorname*{plim}_{n\to\infty} t_{\rm SR} = \sigma_u^{-2}\operatorname{plim} n^{-1/2}\beta(\beta\boldsymbol{y}_1 - \boldsymbol{y}_2)^{\top}\boldsymbol{u},$$

which was the result to be proved.

In order to show that $t_{\rm GNR}$ is asymptotically distributed as $\mathrm{N}(0,1)$ under the null hypothesis, we must show that τ has mean 0 and variance 1 and that it is asymptotically normally distributed. That it has mean 0 follows immediately from the fact that, under the null hypothesis, u_t is uncorrelated with y_{t-1} and y_{t-2}. That it is asymptotically normally distributed follows from the fact that we can apply a central limit theorem to both of the quantities

$$n^{-1/2}\sum_{t=2}^{n}\beta u_t y_{t-1} \quad\text{and}\quad n^{-1/2}\sum_{t=3}^{n}u_t y_{t-2}, \tag{S7.22}$$

since the difference between these quantities, when divided by σ_u^2, is asymptotically equal to τ.

It remains to show that the variance of τ is 1. Since the factors of $n^{-1/2}$ offset the fact that both terms in (S7.22) are summations, it is clear that the variance of τ is $1/\sigma_u^4$ times

$$\operatorname{Var}(\beta u_t y_{t-1} - u_t y_{t-2}) = \mathrm{E}(\beta^2 u_t^2 y_{t-1}^2 - 2\beta u_t^2 y_{t-1} y_{t-2} + u_t^2 y_{t-2}^2). \quad \text{(S7.23)}$$

Because u_t is independent of y_{t-1},

$$\mathrm{E}(u_t^2 y_{t-1}^2) = \sigma_u^2 \mathrm{E}(y_{t-1}^2) = \sigma_u^2 \frac{\sigma_u^2}{1-\beta^2} = \frac{\sigma_u^4}{1-\beta^2}.$$

Similarly, $\mathrm{E}(u_t^2 y_{t-2}^2) = \sigma_u^4/(1-\beta^2)$, and

$$\mathrm{E}(u_t^2 y_{t-1} y_{t-2}) = \sigma_u^2 \mathrm{E}(y_{t-1} y_{t-2}) = \sigma_u^2 \frac{\beta\sigma_u^2}{1-\beta^2} = \frac{\beta\sigma_u^4}{1-\beta^2}.$$

Thus we find that (S7.23) equals

$$\begin{aligned}\frac{\beta^2\sigma_u^4}{1-\beta^2} - 2\frac{\beta^2\sigma_u^4}{1-\beta^2} + \frac{\sigma_u^4}{1-\beta^2} &= \frac{\sigma_u^4}{1-\beta^2} - \frac{\beta^2\sigma_u^4}{1-\beta^2} \\ &= \frac{(1-\beta^2)\sigma_u^4}{1-\beta^2} = \sigma_u^4.\end{aligned}$$

Therefore, the variance of τ itself is just 1, and we conclude that $\tau \sim \mathrm{N}(0,1)$.

Because t_{SR} is asymptotically equal to β times t_{GNR}, it must be asymptotically distributed as $\mathrm{N}(0,\beta^2)$. Since $|\beta| < 1$, the asymptotic variance of t_{SR} is always less than that of t_{GNR}.