Solution to Exercise 6.2

★6.2 Consider a model similar to (3.20), but with error terms that are normally distributed:

$$y_t = \beta_1 + \beta_2 {}^1\!/t + u_t, \quad u_t \sim \mathrm{NID}(0, \sigma^2),$$

where $t = 1, 2, \ldots, n$. If the true value of β_2 is β_2^0 and $\hat{\boldsymbol{\beta}}_2$ is the OLS estimator, show that the limit in probability of $\hat\beta_2 - \beta_2^0$ is a normal random variable with mean 0 and variance $6\sigma^2/\pi^2$. In order to obtain this result, you will need to use the results that

$$\sum_{t=1}^{\infty} (1/t)^2 = \pi^2/6,$$

and that, if $s(n) = \sum_{t=1}^{n}(1/t)$, then $\lim n^{-1}s(n) = 0$ and $\lim n^{-1}s^2(n) = 0$.

Subtracting the sample mean $\bar y$ from y_t and the sample mean $s(n)/n$ from ${}^1\!/t$ is equivalent to regressing each of them on a constant term and taking the residuals. Thus, by the FWL Theorem, the OLS estimator $\hat\beta_2$ is the same as the estimator obtained by regressing $y_t - \bar y$ on ${}^1\!/t - s(n)/n$, which is

$$\hat\beta_2 = \frac{\sum_{t=1}^{n}(y_t - \bar y)\big({}^1\!/t - s(n)/n\big)}{\sum_{t=1}^{n}\big({}^1\!/t - s(n)/n\big)^2}. \tag{S6.01}$$

Since the DGP is assumed to be (3.20) with $\beta_2 = \beta_2^0$,

$$y_t - \bar y = \beta_2^0\big({}^1\!/t - s(n)/n\big) + u_t - \bar u,$$

where $\bar u$ denotes the mean of the u_t. Substituting for $y_t - \bar y$, (S6.01) can be rewritten as

$$\hat\beta_2 - \beta_2^0 = \frac{\sum_{t=1}^{n}(u_t - \bar u)\big({}^1\!/t - s(n)/n\big)}{\sum_{t=1}^{n}\big({}^1\!/t - s(n)/n\big)^2}. \tag{S6.02}$$

It is easy to see from (S6.02) that $\hat\beta_2 - \beta_2^0$ has mean zero. Since ${}^1\!/t - s(n)/n$ is nonstochastic, the expectation of the numerator on the right-hand side is the same as the expectation of $u_t - \bar u$, which is zero. Because the denominator is also nonstochastic, the whole expression must have expectation zero. It must also be normally distributed, because the u_t are assumed to be normally distributed, and a weighted sum of normal random variables is itself normally distributed.

We must now determine the limit of the variance of the right-hand side of equation (S6.02) as $n \to \infty$. Since the mean is zero,

$$\mathrm{Var}(\hat\beta_2 - \beta_2^0) = \mathrm{E}\left(\frac{\sum_{t=1}^{n}(u_t - \bar u)\big({}^1\!/t - s(n)/n\big)}{\sum_{t=1}^{n}\big({}^1\!/t - s(n)/n\big)^2}\right)^2. \tag{S6.03}$$

The numerator of the expression on the right-hand side of this equation is the square of a sum of n terms. Since $\mathrm{E}(u_t u_s) = 0$ for $t \neq s$, $\mathrm{plim}(\bar{u}) = 0$. In addition, $\lim s(n)/n = 0$. Therefore, the expectation of this numerator is asymptotically equivalent to

$$\mathrm{E}\bigg(\sum_{t=1}^{n} u_t^2 ({}^1\!/_t)^2\bigg) = \frac{1}{6}\pi^2\sigma^2. \tag{S6.04}$$

Because $\lim s(n)/n = 0$ and $\lim s^2(n)/n = 0$, the denominator of the expression on the right-hand side of equation (S6.03) is asymptotically equivalent to

$$\bigg(\lim_{n\to\infty} \sum_{t=1}^{n} ({}^1\!/_t)^2\bigg)^2 = \big(\tfrac{1}{6}\pi^2\big)^2. \tag{S6.05}$$

Dividing the right-hand side of (S6.04) by the right-hand side of (S6.05) yields the result that

$$\mathrm{Var}(\hat\beta_2 - \beta_2^0) \overset{a}{=} \frac{6\sigma^2}{\pi^2},$$

which is what we were required to prove.