Solution to Exercise 4.8

⋆4.8 Consider the linear regression model $\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{u}$, where there are n observations and k regressors. Suppose that this model is potentially subject to r restrictions which can be written as $\boldsymbol{R\beta} = \boldsymbol{r}$, where $\boldsymbol{R}$ is an $r \times k$ matrix and $\boldsymbol{r}$ is an r–vector. Rewrite the model so that the restrictions become r zero restrictions.

Since there are (in general) fewer restrictions than there are parameters, we must partition $\boldsymbol{R}$ and $\boldsymbol{\beta}$ so that we can solve for some of the parameters in terms of the others. We therefore begin by rearranging the columns of $\boldsymbol{X}$ so that the restrictions can be written as

$$\boldsymbol{R}_1\boldsymbol{\beta}_1 + \boldsymbol{R}_2\boldsymbol{\beta}_2 = \boldsymbol{r}, \tag{S4.13}$$

where $\boldsymbol{R} \equiv [\boldsymbol{R}_1 \quad \boldsymbol{R}_2]$ and $\boldsymbol{\beta} \equiv [\boldsymbol{\beta}_1 \vdots \boldsymbol{\beta}_2]$, $\boldsymbol{R}_1$ being an $r \times (k-r)$ matrix and $\boldsymbol{R}_2$ being a nonsingular $r \times r$ matrix. It must be possible to do this if the restrictions are in fact distinct. We also partition $\boldsymbol{X}$ as $[\boldsymbol{X}_1 \quad \boldsymbol{X}_2]$, conformably with the partition of $\boldsymbol{\beta}$.

Solving equations (S4.13) for $\boldsymbol{\beta}_2$ yields

$$\boldsymbol{\beta}_2 = \boldsymbol{R}_2^{-1}\boldsymbol{r} - \boldsymbol{R}_2^{-1}\boldsymbol{R}_1\boldsymbol{\beta}_1.$$

Thus the original regression, with the restrictions imposed, can be written as

$$\boldsymbol{y} = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_2(\boldsymbol{R}_2^{-1}\boldsymbol{r} - \boldsymbol{R}_2^{-1}\boldsymbol{R}_1\boldsymbol{\beta}_1) + \boldsymbol{u},$$

This is equivalent to

$$\boldsymbol{y} - \boldsymbol{X}_2\boldsymbol{R}_2^{-1}\boldsymbol{r} = (\boldsymbol{X}_1 - \boldsymbol{X}_2\boldsymbol{R}_2^{-1}\boldsymbol{R}_1)\boldsymbol{\beta}_1 + \boldsymbol{u}. \tag{S4.14}$$

This is a restricted version of the original regression. To obtain a regression equivalent to the original, we have to add back in r regressors that, together with $\boldsymbol{Z}_1 \equiv \boldsymbol{X}_1 - \boldsymbol{X}_2\boldsymbol{R}_2^{-1}\boldsymbol{R}_1$, span the same space as $\boldsymbol{X}$. Although there is an infinite number of ways to do this, the easiest way is to use the r columns of $\boldsymbol{X}_2$ as the additional regressors. To see that this works, note that, for arbitrary $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$,

$$\boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_2\boldsymbol{\beta}_2 = \boldsymbol{Z}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_2(\boldsymbol{\beta}_2 + \boldsymbol{R}_2^{-1}\boldsymbol{R}_1\boldsymbol{\beta}_1),$$

from which it follows that $\mathcal{S}(\boldsymbol{X}_1, \boldsymbol{X}_2) = \mathcal{S}(\boldsymbol{Z}_1, \boldsymbol{X}_2)$. Thus the regression

$$\boldsymbol{y} - \boldsymbol{X}_2\boldsymbol{R}_2^{-1}\boldsymbol{r} = \boldsymbol{Z}_1\boldsymbol{\gamma}_1 + \boldsymbol{X}_2\boldsymbol{\gamma}_2 + \boldsymbol{u}. \tag{S4.15}$$

is equivalent to the original regression, with $\boldsymbol{\beta}_1 = \boldsymbol{\gamma}_1$. In addition, the restrictions that $\boldsymbol{\gamma}_2 = \boldsymbol{0}$ in (S4.15) are equivalent to the restrictions that $\boldsymbol{R\beta} = \boldsymbol{r}$ in the original model.