Solution to Exercise 3.12

$\star$**3.12** Starting from equation (3.42) and using the result proved in Exercise 3.9, but without using (3.43), prove that, if $\mathrm{E}(u_t^2) = \sigma_0^2$ and $\mathrm{E}(u_s u_t) = 0$ for all $s \neq t$, then $\mathrm{Var}(\hat{u}_t) = (1 - h_t)\sigma_0^2$. This is the result (3.44).

The second line of equation (3.42) is

$$\hat{u}_t = u_t - \sum_{s=1}^{n} \boldsymbol{X}_t(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}_s^\top u_s. \tag{S3.11}$$

This expression can be expressed as a sum of mean-zero random variables with zero covariances, as follows:

$$\hat{u}_t = \big(1 - \boldsymbol{X}_t(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}_t^\top\big)u_t - \sum_{s\neq t} \boldsymbol{X}_t(\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}_s^\top u_s.$$

The matrix products here are all elements of the projection matrix $\boldsymbol{P_X}$. To ease notation, denote the ts^{th} element of this matrix by P_{ts}. Then we have

$$\hat{u}_t = (1 - P_{tt})u_t - \sum_{s\neq t} P_{ts}u_s.$$

The variance of $\hat{u}_t$ is the sum of the variances of the terms on the right-hand side of this equation, and so

$$\mathrm{Var}(\hat{u}_t) = \sigma_0^2\Big((1 - P_{tt})^2 + \sum_{s\neq t} P_{ts}^2\Big) = \sigma_0^2\Big(1 - 2P_{tt} + \sum_{s=1}^{n} P_{ts}^2\Big). \tag{S3.12}$$

Since $\boldsymbol{P_X}$ is an orthogonal projection matrix, it is symmetric and idempotent. Therefore

$$\sum_{s=1}^{n} P_{ts}^2 = \sum_{s=1}^{n} P_{ts}P_{st} = \big(\boldsymbol{P}_{\boldsymbol{X}}^2\big)_{tt} = P_{tt}.$$

With this, equation (S3.12) simplifies to

$$\mathrm{Var}(\hat{u}_t) = \sigma_0^2\big(1 - 2P_{tt} + P_{tt}\big) = \sigma_0^2\big(1 - P_{tt}\big) = \sigma_0^2(1 - h_t), \tag{S3.13}$$

which is what we were asked to prove. Note that all expectations are conditional on $\boldsymbol{X}$.