Solution to Exercise 15.9

⋆15.9 Suppose you have a sample of n IID observations u_t, $t = 1, \ldots, n$, on a variable that is supposed to follow the standard normal distribution. What is the very simplest way to test the null hypothesis of normality against alternatives allowing the u_t to be skewed, to have excess kurtosis, or both? Are your proposed tests exact or asymptotic?

Suppose now that the variance of the u_t is unknown and must be estimated from the sample. Are all of the tests you just proposed still valid? If not, explain how some or all of them need to be modified. **Hint:** Recall that the regressor corresponding to σ is asymptotically orthogonal to the test regressor in the OPG regression (15.28).

Because the mean and variance of the u_t are known, there is no need to standardize them. Thus the simplest possible test statistic for skewness is

$$(15n)^{-1/2}\sum_{t=1}^{n} u_t^3 \overset{a}{\sim} \mathrm{N}(0,1). \tag{S15.17}$$

This uses the fact that the sixth moment of the standard normal distribution is 15. Similarly, the simplest possible test statistic for kurtosis is

$$(96n)^{-1/2}\sum_{t=1}^{n}(u_t^4 - 3) \overset{a}{\sim} \mathrm{N}(0,1), \tag{S15.18}$$

because the variance of $u_t^4 - 3$ is

$$\mathrm{E}(u_t^8) - 6\mathrm{E}(u_t^4) + 9 = 105 - 18 + 9 = 96.$$

A valid test statistic for skewness and kurtosis jointly is

$$\Big((15n)^{-1/2}\sum_{t=1}^{n} u_t^3\Big)^2 + \Big((96n)^{-1/2}\sum_{t=1}^{n}(u_t^4 - 3)\Big)^2,$$

which is asymptotically distributed as $\chi^2(2)$.

The three tests just proposed are valid only asymptotically. However, since the statistics themselves are pivotal, we can easily obtain exact inferences by using Monte Carlo tests.

Of course, it would also be valid to estimate the mean and variance of the u_t, standardize them to form e_t, and then use the statistics τ_3, τ_4, and $\tau_{3,4}$ that were described in Section 15.2. The latter procedure might actually work better in practice.

When the variance is unknown, it can be estimated by

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{t=1}^{n} u_t^2.$$

We can formulate tests, as we did in Section 15.2, using an OPG regression. The null hypothesis under test corresponds to the model

$$\boldsymbol{y} = \boldsymbol{u}, \quad \boldsymbol{u} \sim \mathrm{N}(\boldsymbol{0}, \sigma^2\mathbf{I}),$$

and the OPG regression for this model is the following simplified version of regression (15.28):

$$1 = b_\sigma \frac{u_t^2 - \hat{\sigma}^2}{\hat{\sigma}^3} + \text{residual}. \tag{S15.19}$$

If we wish to have a test for skewness only, then the test regressor is u_t^3. As we saw in section 15.2, this test regressor is asymptotically orthogonal to the regressor in (S15.19), and so we can omit the latter from the test regression. This gives

$$1 = cu_t^3 + \text{residual},$$

and, since the regressor and regressand are asymptotically orthogonal under the null, the test statistic is asymptotically equivalent to

$$\frac{n^{-1/2}\sum_{t=1}^n u_t^3}{(n^{-1}\sum_{t=1}^n u_t^6)^{1/2}} = \frac{n^{-1/2}\sum_{t=1}^n e_t^3}{(n^{-1}\sum_{t=1}^n e_t^6)^{1/2}}, \tag{S15.20}$$

where $e_t \equiv u_t/\hat{\sigma}$. If u_t is normal, $\mathrm{E}(u_t^6) = 15\sigma^6$, and $\mathrm{E}\big((u_t/\sigma)^6\big) = 15$. It follows that the rightmost expression in (S15.20) is asymptotically equivalent to $(15n)^{-1/2}\sum_{t=1}^n e_t^3$, which is just (S15.17) with u_t replaced by e_t.

The test for kurtosis is quite different. The test regressor is $u_t^4 - 3\sigma^4$, and, as we have seen, it is not asymptotically orthogonal to the regressor in (S15.19). The statistic (S15.18) is no longer valid, therefore, because we must take explicit account of the parameter uncertainty induced by the use of $\hat{\sigma}^2$ rather than the true variance. As before, however, we may subtract $6\sigma^5$ times the regressor in (S15.19) from the test regressor, so as to make the latter orthogonal to the former. After dividing the result by σ^4, we end up with the test regression (15.32), and, as before, the t statistic from this regression is asymptotically equivalent to the statistic τ_4 of (15.33). A joint test for skewness and kurtosis could use either the statistic $\tau_{3,4}$ defined in equation (15.34), or the statistic

$$\bigg((15n)^{-1/2}\sum_{t=1}^{n} u_t^3\bigg)^2 + \tau_4^2,$$

both of which are asymptotically distributed as $\chi^2(2)$.