Solution to Exercise 15.8

$\star$**15.8** Show that the regressor in the testing regression (15.32) is asymptotically orthogonal to the regressors in the OPG regression (15.27), when all regressors are evaluated at root-n consistent estimators $\acute{\boldsymbol{\beta}}$ and $\acute{s}$. Note that two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are said to be asymptotically orthogonal if $\operatorname{plim} n^{-1}\boldsymbol{a}^\top\boldsymbol{b} = 0$.

Prove that the t statistic from regression (15.32) is asymptotically equivalent to the statistic τ_4 defined by (15.33).

Show also that the statistics τ_3 and τ_4 are asymptotically independent under the null of normality.

The regressor in the testing regression (15.32) is $e_t^4 - 6e_t^2 + 3$, where $e_t = \hat{u}_t/\hat{\sigma}$. The regressors in the OPG regression corresponding to the parameters $\boldsymbol{\beta}$ are the elements of the vector $\acute{u}_t\boldsymbol{X}_t/\acute{s}^2$. Asymptotically, it makes no difference whether we evaluate the residual $u_t(\boldsymbol{\beta})$ at $\hat{\boldsymbol{\beta}}$ or $\acute{\boldsymbol{\beta}}$, because these are both consistent estimates of $\boldsymbol{\beta}$. Similarly, it makes no difference whether we use $\hat{\sigma}$ or $\acute{s}$. In fact, we might just as well evaluate everything at the true values $\boldsymbol{\beta}_0$ and σ_0. Therefore, the probability limit in which we are interested is equal to

$$\operatorname*{plim}_{n\to\infty} \frac{1}{n}(\varepsilon_t^5 - 6\varepsilon_t^3 + 3\varepsilon_t)\boldsymbol{X}_t/\sigma_0, \tag{S15.14}$$

where $\varepsilon_t \equiv u_t(\boldsymbol{\beta}_0)/\sigma_0$ is standard normal. Since the distribution of ε_t is the same conditional on $\boldsymbol{X}_t$ as it is unconditionally, and all odd moments of the normal distribution are zero, we see that

$$\mathrm{E}(\varepsilon_t^5\boldsymbol{X}_t/\sigma_0) = 6\mathrm{E}(\varepsilon_t^3\boldsymbol{X}_t/\sigma_0) = 3\mathrm{E}(\varepsilon_t\boldsymbol{X}_t/\sigma_0) = 0.$$

Because all the moments of the normal distribution are finite, we can apply a law of large numbers to each of the three terms in expression (S15.14). Therefore, we conclude that this probability limit is indeed zero, as we were required to show.

The second regressor in the OPG regression is $(1/\acute{s}^3)(\acute{u}_t^2 - \acute{s}^2)$. Once again, the consistency of all the estimators implies that we can evaluate everything at the true parameter values, so that the plim in which we are interested is

$$\begin{aligned} &\operatorname*{plim}_{n\to\infty} \frac{1}{n}(\varepsilon_t^4 - 6\varepsilon_t^2 + 3)(\varepsilon_t^2 - 1)/\sigma_0 \\ &\quad = \operatorname*{plim}_{n\to\infty} \frac{1}{n}(\varepsilon_t^6 - 7\varepsilon_t^4 + 9\varepsilon_t^2 - 3)/\sigma_0. \end{aligned} \tag{S15.15}$$

From the result of Exercise 13.19, we see that $\mathrm{E}(\varepsilon_t^6) = 15$ and that $\mathrm{E}(\varepsilon_t^4) = 3$. Therefore,

$$\mathrm{E}(\varepsilon_t^6 - 7\varepsilon_t^4 + 9\varepsilon_t^2 - 3) = 15 - 21 + 9 - 3 = 0.$$

Applying a law of large numbers to the right-hand side of equation (S15.15) then allows us to conclude that this probability limit is zero.

We now turn our attention to the t statistic for $c = 0$ in the simple testing regression (15.32). This regression should have no explanatory power, asymptotically under the null hypothesis. Therefore, we can ignore the fact that the t statistic depends on the standard error of the regression and simply replace the latter with 1. The numerator of the t statistic is

$$n^{-1/2}\sum_{t=1}^{n}(e_t^4 - 6e_t^2 + 3).$$

Since the e_t^2 sum to n, this is equal to

$$n^{-1/2}\sum_{t=1}^{n}(e_t^4 - 3). \tag{S15.16}$$

The denominator of the t statistic is the square root of

$$\frac{1}{n}\sum_{t=1}^{n}(e_t^4 - 6e_t^2 + 3)^2 = \frac{1}{n}\sum_{t=1}^{n}(e_t^8 - 12e_t^6 + 42e_t^4 - 36e_t^2 + 9).$$

Using once more the result of Exercise 13.19, we see that the expectation of the term inside the parentheses is

$$105 - 180 + 126 - 36 + 9 = 24.$$

Therefore, the t statistic is asymptotically equivalent to expression (S15.16) divided by the square root of 24, which is

$$\frac{n^{-1/2}}{24^{1/2}}\sum_{t=1}^{n}(e_t^4 - 3) = (24n)^{-1/2}\sum_{t=1}^{n}(e_t^4 - 3).$$

This is the test statistic τ_4 defined in equation (15.33).

Finally, we wish to show that the test statistics τ_3 and τ_4 are asymptotically independent under the null hypothesis. Each of these statistics is asymptotically normally distributed and equal to a summation, which is random, times a scaling factor. Therefore, all we need to show is that the covariance of the two summations is zero. Since the terms corresponding to different observations are independent by hypothesis, it is enough to show that the covariances are zero observation by observation. Moreover, we can replace e_t by ε_t in both of the summations, since $e_t \to \varepsilon_t$ as $n \to \infty$. Thus we wish to show that

$$\mathrm{E}\big((\varepsilon_t^4 - 3)\varepsilon_t^3\big) = \mathrm{E}(\varepsilon_t^7) - 3\mathrm{E}(\varepsilon_t^3) = 0.$$

This result follows at once from the fact that the odd moments of the normal distribution are zero.