# Solution to Exercise 15.4

**⋆15.4** Let (15.01) be an artificial regression corresponding to a model $\mathbb{M}$ and an asymptotically normal root-$n$ consistent estimator $\hat{\boldsymbol{\theta}}$ of the parameters of $\mathbb{M}$, with the asymptotic covariance matrix of $\hat{\boldsymbol{\theta}}$ given by (15.03). Show that, whenever $\acute{\boldsymbol{\theta}}$ is a root-$n$ consistent estimator, $r$ times the $F$ statistic for the artificial hypothesis that $\boldsymbol{c} = \boldsymbol{0}$ in the artificial regression (15.05) is asymptotically distributed as $\chi^2(r)$ under any DGP in $\mathbb{M}$.

If we make the definitions $\acute{\boldsymbol{r}} \equiv \boldsymbol{r}(\acute{\boldsymbol{\theta}})$, $\acute{\boldsymbol{R}} \equiv \boldsymbol{R}(\acute{\boldsymbol{\theta}})$, and $\acute{\boldsymbol{Z}} \equiv \boldsymbol{Z}(\acute{\boldsymbol{\theta}})$, the artificial regression (15.05) can be written as

$$\acute{\boldsymbol{r}} = \acute{\boldsymbol{R}}\boldsymbol{b} + \acute{\boldsymbol{Z}}\boldsymbol{c} + \text{residuals}.$$

The $F$ statistic for $\boldsymbol{c} = \boldsymbol{0}$ in this regression is

$$\frac{\acute{\boldsymbol{r}}^\top \boldsymbol{M}_{\acute{\boldsymbol{R}}}\acute{\boldsymbol{Z}}(\acute{\boldsymbol{Z}}^\top \boldsymbol{M}_{\acute{\boldsymbol{R}}}\acute{\boldsymbol{Z}})^{-1}\acute{\boldsymbol{Z}}^\top \boldsymbol{M}_{\acute{\boldsymbol{R}}}\acute{\boldsymbol{r}}/r}{\acute{\boldsymbol{r}}^\top \boldsymbol{M}_{\acute{\boldsymbol{R}},\acute{\boldsymbol{Z}}}\acute{\boldsymbol{r}}/(n-k-r)}. \tag{S15.05}$$

Notice that $r$ times the numerator of this test statistic is expression (15.79). As was shown in Section 15.7, this is asymptotically equal to expression (15.85), which is rewritten here for convenience:

$$(n^{-1/2}\boldsymbol{r}_0^\top \boldsymbol{M}_{\boldsymbol{R}_0}\boldsymbol{Z}_0)(n^{-1}\boldsymbol{Z}_0^\top \boldsymbol{M}_{\boldsymbol{R}_0}\boldsymbol{Z}_0)^{-1}(n^{-1/2}\boldsymbol{Z}_0^\top \boldsymbol{M}_{\boldsymbol{R}_0}\boldsymbol{r}_0). \tag{15.85}$$

This is a quadratic form in the $r$–vector $n^{-1/2}\boldsymbol{Z}_0^\top \boldsymbol{M}_{\boldsymbol{R}_0}\boldsymbol{r}_0$ and a matrix which, as will show in a moment, is proportional to the inverse of its asymptotic covariance matrix.

The asymptotic covariance matrix of the vector $n^{-1/2}\boldsymbol{Z}_0^\top \boldsymbol{M}_{\boldsymbol{R}_0}\boldsymbol{r}_0$ is

$$\mathrm{Var}\big(\operatorname*{plim}_{n\to\infty} n^{-1/2}\boldsymbol{Z}_0^\top \boldsymbol{M}_{\boldsymbol{R}_0}\boldsymbol{r}_0\big) = \operatorname*{plim}_{n\to\infty} \frac{1}{n}\boldsymbol{Z}_0^\top \boldsymbol{M}_{\boldsymbol{R}_0}\boldsymbol{r}_0\boldsymbol{r}_0^\top \boldsymbol{M}_{\boldsymbol{R}_0}\boldsymbol{Z}_0.$$

By essentially the same argument as the one that led to equation (15.84), this covariance matrix is equal to

$$\sigma_0^2 \operatorname*{plim}_{n\to\infty} \frac{1}{n}\boldsymbol{Z}_0^\top \boldsymbol{M}_{\boldsymbol{R}_0}\boldsymbol{Z}_0, \tag{S15.06}$$

where $\sigma_0^2$ is the plim of the OLS estimate of the error variance from the artificial regression. There is only one difference between the argument that led to equation (15.84) and the one that leads to equation (S15.06). Where we previously used the version of condition R2 that gives

$$\operatorname*{plim}_{n\to\infty} \frac{1}{n}\boldsymbol{R}_0^\top \boldsymbol{r}_0\boldsymbol{r}_0^\top \boldsymbol{R}_0 = \operatorname*{plim}_{n\to\infty} \frac{1}{n}\boldsymbol{R}_0^\top \boldsymbol{R}_0,$$

where either or both instances of $\boldsymbol{R}_0$ may be replaced by $\boldsymbol{Z}_0$, we now use the version that gives

$$\operatorname*{plim}_{n\to\infty} \frac{1}{n} \boldsymbol{R}_0^{\top} \boldsymbol{r}_0 \boldsymbol{r}_0^{\top} \boldsymbol{R}_0 = \sigma_0^2 \operatorname*{plim}_{n\to\infty} \frac{1}{n} \boldsymbol{R}_0^{\top} \boldsymbol{R}_0.$$

Thus we see that expression (15.85) is equal to $\sigma_0^2$ times a quadratic form in the $r$–vector $n^{-1/2}\boldsymbol{Z}_0^{\top}\boldsymbol{M}_{\boldsymbol{R}_0}\boldsymbol{r}_0$ and the inverse of its asymptotic covariance matrix. Under standard regularity conditions, we can apply a CLT to the vector $n^{-1/2}\boldsymbol{Z}_0^{\top}\boldsymbol{M}_{\boldsymbol{R}_0}\boldsymbol{r}_0$ to conclude that it is asymptotically normally distributed. Therefore, by Theorem 4.1, this quadratic form follows the $\chi^2(r)$ distribution asymptotically.

It remains to be shown that the denominator of the $F$ statistic (S15.05) has a plim of $\sigma_0^2$. This denominator must then cancel with the factor of $\sigma_0^2$ in the numerator, implying that $r$ times the $F$ statistic itself follows the $\chi^2(r)$ distribution asymptotically. By the consistency of $\acute{\boldsymbol{\theta}}$,

$$\frac{\acute{\boldsymbol{r}}^{\top}\boldsymbol{M}_{\acute{\boldsymbol{R}},\acute{\boldsymbol{Z}}}\acute{\boldsymbol{r}}}{n-k-r} \overset{a}{=} \frac{\boldsymbol{r}_0^{\top}\boldsymbol{M}_{\boldsymbol{R}_0,\boldsymbol{Z}_0}\boldsymbol{r}_0}{n-k-r}. \tag{S15.07}$$

Neither $\boldsymbol{R}_0$ nor $\boldsymbol{Z}_0$ can have any explanatory power for $\boldsymbol{r}_0$ asymptotically, the former because the first condition for an artificial regression to be valid together with the consistency of $\hat{\boldsymbol{\theta}}$ implies it, and the latter because it was explicitly assumed in condition R1. Therefore, the plim of the right-hand side of equation (S15.07) must be the same as the plim of $n^{-1}\boldsymbol{r}_0^{\top}\boldsymbol{r}_0$, which is $\sigma_0^2$. This completes the proof.