Solution to Exercise 15.18

⋆15.18 Consider the two nonnested linear regression models H_1 and H_2 given in equations (15.37). An encompassing test can be based on the estimate of the error variance of model H_2 rather than on the estimates of the parameters $\boldsymbol{\gamma}$. Let $\hat{\sigma}_2^2$ be the usual ML estimator obtained from estimating H_2. Compute the expectation of $\hat{\sigma}_2^2$ under the DGP in model H_1 with parameters $\boldsymbol{\beta}$ and σ_1^2. Explain how to estimate this expectation based on the parameter estimates for model H_1. Let $\tilde{\sigma}_2^2$ denote the consistent estimator you are proposing.

Show that $n^{1/2}(\hat{\sigma}_2^2 - \tilde{\sigma}_2^2)$ is asymptotically equal to a random variable that is proportional to $\boldsymbol{u}^\top \boldsymbol{M_X P_Z X}\boldsymbol{\beta}_0$, in the notation of equation (15.41). What does this result imply about the relationship between the variance encompassing test and the J test?

By the usual formula for the ML estimate of the variance of a linear regression model estimated by OLS,

$$\hat{\sigma}_2^2 = \frac{1}{n}\boldsymbol{y}^\top \boldsymbol{M_Z y}.$$

Under the specified DGP, $\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta}_0 + \boldsymbol{u}$. Therefore,

$$\hat{\sigma}_2^2 = \frac{1}{n}(\boldsymbol{X}\boldsymbol{\beta}_0 + \boldsymbol{u})^\top \boldsymbol{M_Z}(\boldsymbol{X}\boldsymbol{\beta}_0 + \boldsymbol{u}).$$

If we expand the quadratic form, there are four terms, but two of them have expectation 0. Thus we find that

$$\begin{aligned}
\mathrm{E}(\hat{\sigma}_2^2) &= \frac{1}{n}\mathrm{E}(\boldsymbol{\beta}_0^\top \boldsymbol{X}^\top \boldsymbol{M_Z X}\boldsymbol{\beta}_0) + \frac{1}{n}\mathrm{E}(\boldsymbol{u}^\top \boldsymbol{M_Z u}) \\
&= \frac{1}{n}\boldsymbol{\beta}_0^\top \boldsymbol{X}^\top \boldsymbol{M_Z X}\boldsymbol{\beta}_0 + \frac{n-k_2}{n}\sigma_1^2. \qquad \text{(S15.35)}
\end{aligned}$$

In dropping the expectations operator from the first term, we are implicitly assuming either that $\boldsymbol{X}$ and $\boldsymbol{Z}$ are fixed or that we are taking expectations conditional on an information set to which they belong. Asymptotically, it makes no difference whether or not we retain the expectations operator.

The natural estimator of $\mathrm{E}(\hat{\sigma}_2^2)$ is

$$\tilde{\sigma}_2^2 = \hat{\sigma}_1^2 + \frac{1}{n}\hat{\boldsymbol{\beta}}^\top \boldsymbol{X}^\top \boldsymbol{M_Z X}\hat{\boldsymbol{\beta}},$$

where we have ignored the factor of $(n-k_2)/n$, which is irrelevant asymptotically. Since $\boldsymbol{X}\hat{\boldsymbol{\beta}} = \boldsymbol{P_X y}$, this can be rewritten as

$$\tilde{\sigma}_2^2 = \frac{1}{n}\boldsymbol{y}^\top \boldsymbol{M_X y} + \frac{1}{n}\boldsymbol{y}^\top \boldsymbol{P_X M_Z P_X y}.$$

We now consider the quantity $n^{1/2}(\hat{\sigma}_2^2 - \tilde{\sigma}_2^2)$, which is equal to

$$n^{-1/2}(\boldsymbol{y}^\top \boldsymbol{M_Z y} - \boldsymbol{y}^\top \boldsymbol{M_X y} - \boldsymbol{y}^\top \boldsymbol{P_X M_Z P_X y}).$$

If we replace $\boldsymbol{y}$ by $\boldsymbol{X\beta}_0+\boldsymbol{u}$, the three terms inside the parentheses here become

$$\boldsymbol{\beta}_0^\top\boldsymbol{X}^\top\boldsymbol{M_Z X\beta}_0 + \boldsymbol{u}^\top\boldsymbol{M_Z u} + 2\boldsymbol{u}^\top\boldsymbol{M_Z X\beta}_0 - \boldsymbol{u}^\top\boldsymbol{M_X u} - \boldsymbol{\beta}_0^\top\boldsymbol{X}^\top\boldsymbol{M_Z X\beta}_0$$
$$- 2\boldsymbol{u}^\top\boldsymbol{P_X M_Z X\beta}_0 - \boldsymbol{u}^\top\boldsymbol{P_X M_Z P_X u},$$

where we have used the facts that $\boldsymbol{M_X}$ annihilates $\boldsymbol{X\beta}_0$, that $\boldsymbol{P_X X\beta}_0 = \boldsymbol{X\beta}_0$, and that a scalar is equal to its transpose. The first term here cancels with the fifth. The third and sixth terms can be combined, as follows:

$$2\boldsymbol{u}^\top\boldsymbol{M_Z X\beta}_0 - 2\boldsymbol{u}^\top\boldsymbol{P_X M_Z X\beta}_0 = 2\boldsymbol{u}^\top\boldsymbol{M_X M_Z X\beta}_0 = -2\boldsymbol{u}^\top\boldsymbol{M_X P_Z X\beta}_0.$$

Therefore, $n^{1/2}(\hat\sigma_2^2 - \tilde\sigma_2^2)$ is equal to

$$-2\boldsymbol{u}^\top\boldsymbol{M_X P_Z X\beta}_0 + \boldsymbol{u}^\top\boldsymbol{M_Z u} - \boldsymbol{u}^\top\boldsymbol{M_X u} - \boldsymbol{u}^\top\boldsymbol{P_X M_Z P_X u}.$$

We need to show that the first term here, which is evidently $O_p(n^{1/2})$, is the term of leading order. The remaining three terms are

$$\begin{aligned}
&\boldsymbol{u}^\top\boldsymbol{M_Z u} - \boldsymbol{u}^\top\boldsymbol{M_X u} - \boldsymbol{u}^\top\boldsymbol{P_X M_Z P_X u}\\
&\quad = \boldsymbol{u}^\top\boldsymbol{M_Z u} - \boldsymbol{u}^\top\boldsymbol{M_X u} - \boldsymbol{u}^\top\boldsymbol{P_X u} + \boldsymbol{u}^\top\boldsymbol{P_X P_Z P_X u}\\
&\quad = \boldsymbol{u}^\top\boldsymbol{M_Z u} - \boldsymbol{u}^\top\boldsymbol{u} + \boldsymbol{u}^\top\boldsymbol{P_X P_Z P_X u}\\
&\quad = -\boldsymbol{u}^\top\boldsymbol{P_Z u} + \boldsymbol{u}^\top\boldsymbol{P_X P_Z P_X u}.
\end{aligned}$$

We saw in the answer to Exercise 15.12 that both terms in the last line here are $O_p(1)$. Thus we conclude that

$$n^{1/2}(\hat\sigma_2^2 - \tilde\sigma_2^2) \stackrel{a}{=} -2n^{-1/2}\boldsymbol{u}^\top\boldsymbol{M_X P_Z X\beta}_0.$$

This is what we set out to show. Notice that the right-hand side of this equation is proportional, with factor of proportionality $-2n^{-1/2}$, to $\boldsymbol{u}^\top\boldsymbol{M_X P_Z X\beta}_0$, which is what the numerator of the J statistic is equal to asymptotically. Since the factor of proportionality vanishes when we compute a test statistic based on $n^{1/2}(\hat\sigma_2^2 - \tilde\sigma_2^2)$, we conclude that the variance encompassing statistic and the J statistic are asymptotically equivalent under the null hypothesis.