Solution to Exercise 15.12

⋆15.12 Show that the J statistic computed using regression (15.40) is given by

$$J = \frac{(n-k_1-1)^{1/2}\boldsymbol{y}^\top\boldsymbol{M_X P_Z y}}{(\boldsymbol{y}^\top\boldsymbol{M_X y}\,\boldsymbol{y}^\top\boldsymbol{P_Z M_X P_Z y} - (\boldsymbol{y}^\top\boldsymbol{M_X P_Z y})^2)^{1/2}}.$$

Use this expression to show that the probability limit under hypothesis H_1 of n^{-1} times the square of the denominator is

$$\sigma_0^2 \operatorname*{plim}_{n\to\infty} \frac{1}{n}\boldsymbol{\beta}_0{}^\top\boldsymbol{X}^\top\boldsymbol{P_Z M_X P_Z X}\boldsymbol{\beta}_0,$$

where $\boldsymbol{\beta}_0$ and σ_0^2 are the true parameters.

The FWL regression that corresponds to regression (15.40) is

$$\boldsymbol{M_X y} = \alpha\boldsymbol{M_X P_Z y} + \text{residuals}. \tag{S15.26}$$

Therefore, the t statistic for $\alpha = 0$ is

$$\frac{\boldsymbol{y}^\top\boldsymbol{M_X P_Z y}}{s(\boldsymbol{y}^\top\boldsymbol{P_Z M_X P_Z y})^{1/2}}, \tag{S15.27}$$

where s denotes the standard error of regression for the test regression (15.40). The square of this standard error is

$$s^2 = \frac{\boldsymbol{y}^\top\boldsymbol{M}_{\boldsymbol{X},\boldsymbol{P_Z y}}\boldsymbol{y}}{n-k_1-1}.$$

In order to show that the t statistic (S15.27) is equal to the expression for J given in the exercise, we use the fact that $\boldsymbol{M}_{\boldsymbol{X},\boldsymbol{P_Z y}} = \boldsymbol{M_X} - \boldsymbol{P}_{\boldsymbol{M_X P_Z y}}$ so as to rewrite the numerator of s^2 as

$$\boldsymbol{y}^\top\boldsymbol{M_X y} - \boldsymbol{y}^\top\boldsymbol{M_X P_Z y}(\boldsymbol{y}^\top\boldsymbol{P_Z M_X P_Z y})^{-1}\boldsymbol{y}^\top\boldsymbol{P_Z M_X y}.$$

This is just the SSR from OLS estimation of the H_1 model minus the ESS from the FWL regression (S15.26). Then we see that the denominator of the t statistic (S15.27) is the square root of

$$\big(\boldsymbol{y}^\top\boldsymbol{M_X y} - \boldsymbol{y}^\top\boldsymbol{M_X P_Z y}(\boldsymbol{y}^\top\boldsymbol{P_Z M_X P_Z y})^{-1}\boldsymbol{y}^\top\boldsymbol{P_Z M_X y}\big)\boldsymbol{y}^\top\boldsymbol{P_Z M_X P_Z y}$$

divided by $n-k_1-1$. Since each of the quadratic forms in $\boldsymbol{y}$ here is a scalar, this expression is equal to

$$\boldsymbol{y}^\top\boldsymbol{M_X y}\,\boldsymbol{y}^\top\boldsymbol{P_Z M_X P_Z y} - (\boldsymbol{y}^\top\boldsymbol{M_X P_Z y})^2, \tag{S15.28}$$

the square root of which is the denominator of the expression for J given in the exercise. The factor of the square root of $n-k_1-1$, which implicitly appears in the denominator of the denominator of expression (S15.27), together with the numerator of (S15.27), give the numerator of the expression for J, the validity of which is therefore proved.

We now calculate the plim of n^{-1} times expression (S15.28) under H_1. In order to do this, we need to evaluate the probability limits of three expressions. The easiest of these to deal with is $\boldsymbol{y}^\top\boldsymbol{M_X}\boldsymbol{y}$. Since $\boldsymbol{M_X}$ annihilates $\boldsymbol{X\beta}_0$, the plim of n^{-1} times this expression is just

$$\operatorname*{plim}_{n\to\infty}\frac{1}{n}\boldsymbol{u}^\top\boldsymbol{M_X}\boldsymbol{u}=\sigma_0^2,$$

by standard results from Section 3.6. The next easiest is $\boldsymbol{y}^\top\boldsymbol{M_X}\boldsymbol{P_Z}\boldsymbol{y}$. The plim of n^{-1} times this expression is

$$\operatorname*{plim}_{n\to\infty}\frac{1}{n}\boldsymbol{u}^\top\boldsymbol{M_X}\boldsymbol{P_Z}\boldsymbol{X\beta}_0+\operatorname*{plim}_{n\to\infty}\frac{1}{n}\boldsymbol{u}^\top\boldsymbol{M_X}\boldsymbol{P_Z}\boldsymbol{u}.$$

The first term here has a plim of 0 whenever $\boldsymbol{u}$ is asymptotically orthogonal to $\boldsymbol{X}$ and $\boldsymbol{Z}$. The second term must also have a plim of 0, because, as we saw in Section 15.3, the numerator is $O_p(1)$. Finally, we come to the expression $\boldsymbol{y}^\top\boldsymbol{P_Z}\boldsymbol{M_X}\boldsymbol{P_Z}\boldsymbol{y}$. The plim of n^{-1} times this expression is

$$\begin{aligned}&\operatorname*{plim}_{n\to\infty}\frac{1}{n}\boldsymbol{\beta}_0^\top\boldsymbol{X}^\top\boldsymbol{P_Z}\boldsymbol{M_X}\boldsymbol{P_Z}\boldsymbol{X\beta}_0+\operatorname*{plim}_{n\to\infty}\frac{1}{n}\boldsymbol{u}^\top\boldsymbol{P_Z}\boldsymbol{M_X}\boldsymbol{P_Z}\boldsymbol{u}\\&\quad+\operatorname*{plim}_{n\to\infty}\frac{1}{n}\boldsymbol{\beta}_0^\top\boldsymbol{X}^\top\boldsymbol{P_Z}\boldsymbol{M_X}\boldsymbol{P_Z}\boldsymbol{u}+\operatorname*{plim}_{n\to\infty}\frac{1}{n}\boldsymbol{u}^\top\boldsymbol{P_Z}\boldsymbol{M_X}\boldsymbol{P_Z}\boldsymbol{X\beta}_0.\end{aligned}\tag{S15.29}$$

The first term here requires no further analysis. The third and fourth terms evidently have plims of 0 whenever $\boldsymbol{u}$ is asymptotically orthogonal to $\boldsymbol{X}$ and $\boldsymbol{Z}$. That leaves only the second term, which is equal to

$$\operatorname*{plim}_{n\to\infty}\frac{1}{n}\boldsymbol{u}^\top\boldsymbol{P_Z}\boldsymbol{u}-\operatorname*{plim}_{n\to\infty}\frac{1}{n}\boldsymbol{u}^\top\boldsymbol{P_Z}\boldsymbol{P_X}\boldsymbol{P_Z}\boldsymbol{u}.\tag{S15.30}$$

Consider the first of these two terms. By part 2 of Theorem 4.1, $\boldsymbol{u}^\top\boldsymbol{P_Z}\boldsymbol{u}$ is distributed as $\chi^2(k_2)$ if the error terms are normal. If not, then it is still asymptotically distributed as $\chi^2(k_2)$. In either case, the expression is $O_p(1)$, and so, because the denominator is n, the first term of (S15.30) is equal to 0. A very similar argument applies to the second term. This time, there are seven factors instead of three, and all of them are once again $O_p(1)$, for exactly the same reasons. Thus we see that expression (S15.30) is equal to 0. Therefore, the only term in expression (S15.29) that is not equal to zero is the first term.

Combining these results, we find that

$$\begin{aligned}&\operatorname*{plim}_{n\to\infty}\frac{1}{n}\big(\boldsymbol{y}^\top\boldsymbol{M_X}\boldsymbol{y}\,\boldsymbol{y}^\top\boldsymbol{P_Z}\boldsymbol{M_X}\boldsymbol{P_Z}\boldsymbol{y}-(\boldsymbol{y}^\top\boldsymbol{M_X}\boldsymbol{P_Z}\boldsymbol{y})^2\big)\\&\qquad=\sigma_0^2\operatorname*{plim}_{n\to\infty}\frac{1}{n}\boldsymbol{\beta}_0^\top\boldsymbol{X}^\top\boldsymbol{P_Z}\boldsymbol{M_X}\boldsymbol{P_Z}\boldsymbol{X\beta}_0,\end{aligned}$$

which is what we set out to show.