Solution to Exercise 15.11

★15.11 Consider the classical linear regression model

$$y_t = \beta_1 + \beta_2 x_{t2} + \beta_3 x_{t3} + u_t, \quad u_t \sim \text{NID}(0, \sigma^2),$$

where x_{t2} and x_{t3} are exogenous variables, and there are n observations. Write down the contribution to the loglikelihood made by the t^{th} observation. Then calculate the matrix $\boldsymbol{M}(\hat{\boldsymbol{\theta}})$ of which the typical element is expression (15.36) evaluated at the ML estimates. How many columns does this matrix have? What is a typical element of each of the columns?

Explain how to compute an information matrix test for this model using the OPG regression (15.25). How many regressors does the test regression have? What test statistic would you use, and how many degrees of freedom does it have? What types of misspecification is this test sensitive to?

For this model, the contribution to the loglikelihood for observation t is

$$\ell_t(\boldsymbol{\beta}, \sigma) = -\tfrac{1}{2}\log(2\pi) - \log(\sigma) - \frac{1}{2\sigma^2}(y_t - \beta_1 - \beta_2 x_{t2} - \beta_3 x_{t3})^2.$$

The first derivatives of $\ell_t(\boldsymbol{\beta}, \sigma)$ are

$$\begin{aligned}
\frac{\partial \ell_t}{\partial \beta_1} &= \frac{1}{\sigma^2}(y_t - \beta_1 - \beta_2 x_{t2} - \beta_3 x_{t3}),\\
\frac{\partial \ell_t}{\partial \beta_2} &= \frac{1}{\sigma^2}(y_t - \beta_1 - \beta_2 x_{t2} - \beta_3 x_{t3})x_{t2},\\
\frac{\partial \ell_t}{\partial \beta_3} &= \frac{1}{\sigma^2}(y_t - \beta_1 - \beta_2 x_{t2} - \beta_3 x_{t3})x_{t3}, \text{ and}\\
\frac{\partial \ell_t}{\partial \sigma} &= -\frac{1}{\sigma} + \frac{1}{\sigma^3}(y_t - \beta_1 - \beta_2 x_{t2} - \beta_3 x_{t3})^2.
\end{aligned} \tag{S15.21}$$

Let $x_{t1} = 1$ for all t. Then the second derivatives of $\ell_t(\boldsymbol{\beta}, \sigma)$ are

$$\begin{aligned}
\frac{\partial^2 \ell_t}{\partial \beta_i \partial \beta_j} &= -\frac{1}{\sigma^2} x_{ti} x_{tj}, \text{ for } i, j = 1, \ldots, 3,\\
\frac{\partial^2 \ell_t}{\partial \beta_i \partial \sigma} &= -\frac{2}{\sigma^3}(y_t - \beta_1 - \beta_2 x_{t2} - \beta_3 x_{t3})x_{ti}, \text{ for } i = 1, \ldots, 3, \text{ and}\\
\frac{\partial^2 \ell_t}{\partial \sigma^2} &= \frac{1}{\sigma^2} - \frac{3}{\sigma^4}(y_t - \beta_1 - \beta_2 x_{t2} - \beta_3 x_{t3})^2.
\end{aligned}$$

The matrix $\hat{\boldsymbol{M}}$ has $\frac{1}{2}k(k+1)$ columns, where k is the number of parameters. Since $k = 4$ here, the matrix $\hat{\boldsymbol{M}}$ has 10 columns, although, as we will see, one of these must be dropped because it is perfectly collinear with one of the columns of $\hat{\boldsymbol{G}}$.

Let $\hat{u}_t$ denote the t^{th} OLS residual, and let $\hat{\sigma}$ denote the ML estimate of σ. Then the six columns of $\hat{\boldsymbol{M}}$ that correspond to β_i and β_j for $i, j = 1, \ldots, 3$, $i \leq j$, have typical elements

$$\frac{1}{\hat{\sigma}^4}\,\hat{u}_t^2 x_{ti} x_{tj} - \frac{1}{\hat{\sigma}^2}\,x_{ti} x_{tj}. \tag{S15.22}$$

The three columns of $\hat{\boldsymbol{M}}$ that correspond to β_i and σ for $i = 1, \ldots, 3$ have typical elements

$$\frac{1}{\hat{\sigma}^5}\,\hat{u}_t^3 x_{ti} - \frac{3}{\hat{\sigma}^3}\,\hat{u}_t x_{ti}. \tag{S15.23}$$

Finally, the column of $\hat{\boldsymbol{M}}$ that corresponds to σ alone has typical element

$$\frac{2}{\hat{\sigma}^2} + \frac{1}{\hat{\sigma}^6}\,\hat{u}_t^4 - \frac{5}{\hat{\sigma}^4}\,\hat{u}_t^2. \tag{S15.24}$$

To set up the OPG testing regression (15.25), we evaluate all the variables at the ML estimates $\hat{\boldsymbol{\beta}}$ and $\hat{\sigma}$. For the matrix $\boldsymbol{G}(\hat{\boldsymbol{\theta}})$, we use regressors of which the typical elements are the expressions on the right-hand sides of equations (S15.21). These are the same as the regressors of the OPG regression (15.28) that corresponds to the linear regression model. For the matrix $\boldsymbol{M}(\hat{\boldsymbol{\theta}})$, we use all but one of the columns of the matrix $\hat{\boldsymbol{M}}$.

We do not use all of the columns, because, as noted above, one of the columns of $\hat{\boldsymbol{M}}$ is collinear with one of the columns of $\boldsymbol{G}(\hat{\boldsymbol{\theta}})$. In fact, the column of $\hat{\boldsymbol{M}}$ that corresponds to β_1 alone can be written, using expression (S15.22), as

$$\frac{1}{\hat{\sigma}^4}\,\hat{u}_t^2 - \frac{1}{\hat{\sigma}^2}. \tag{S15.25}$$

This is perfectly collinear with the column of $\hat{\boldsymbol{G}}$ that corresponds to σ, which, from (S15.21), has typical element

$$-\frac{1}{\hat{\sigma}} + \frac{1}{\hat{\sigma}^3}\,\hat{u}_t^2.$$

It is evident that $1/\hat{\sigma}$ times this expression is equal to expression (S15.25). Thus there are only 9 test regressors in the OPG testing regression, and the test statistic, of which the simplest form is just the explained sum of squares from the testing regression, has 9 degrees of freedom.

Asymptotically, we would expect n^{-1} times the inner products of the nine columns that have typical elements given by expressions (S15.22) through (S15.24) with a vector of 1s to converge to the expectation of each of these expressions when we replace $\hat{u}_t$ by u_t and $\hat{\sigma}$ by σ. Therefore, finding these expectations should indicate what types of misspecification the information matrix test is sensitive to.

The column with typical element (S15.24) is testing for excess kurtosis. Under normality, if we were to replace $\hat{u}_t$ by u_t and $\hat{\sigma}$ by σ, this expression would have expectation 0. But it would not have expectation 0 if the fourth moment of u_t were not equal to $3\sigma^4$.

The four columns with typical element (S15.23) are testing for skewness and skewness interacted with the regressors. For a symmetric distribution, u_t^3 must have expectation 0. If this is true conditionally on the x_{ti} (which implies that it is also true unconditionally, because one of the x_{ti} is a constant), then expression (S15.25) with $\hat{u}_t$ replaced by u_t and $\hat{\sigma}$ replaced by σ would have expectation 0.

The five columns with typical element (S15.22) are testing for heteroskedasticity. Expression (S15.22), with $\hat{u}_t$ replaced by u_t and $\hat{\sigma}$ replaced by σ, must have expectation 0 whenever the expectation of u_t^2 conditional on $x_{ti}x_{tj}$ is the same as its unconditional expectation of σ^2.