Solution to Exercise 14.24

⋆14.24 Let $\boldsymbol{A} \equiv [\boldsymbol{a}_1 \;\; \boldsymbol{a}_2]$ be an $n \times 2$ matrix, and let θ be the angle between the nonzero vectors $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$. Show that the columns of the matrix

$$\boldsymbol{AB} \equiv \boldsymbol{A} \begin{bmatrix} \|\boldsymbol{a}_1\|^{-1} & -\|\boldsymbol{a}_1\|^{-1} \cot\theta \\ 0 & \|\boldsymbol{a}_2\|^{-1} \operatorname{cosec}\theta \end{bmatrix}$$

are orthonormal. Use this result to show that the determinant of the 2×2 matrix $\boldsymbol{A}^\top\!\boldsymbol{A}$ is equal to $\|\boldsymbol{a}_1\|^2 \|\boldsymbol{M}_1\boldsymbol{a}_2\|^2$, where $\boldsymbol{M}_1$ is the orthogonal projection on to $\mathcal{S}^\perp(\boldsymbol{a}_1)$.

Let $\boldsymbol{v}$ be an n–vector, and let $\boldsymbol{M_v}$ project orthogonally on to $\mathcal{S}^\perp(\boldsymbol{v})$. Show that the determinant of the 2×2 matrix $\boldsymbol{A}^\top\boldsymbol{M_v}\boldsymbol{A}$ is equal to the determinant of $\boldsymbol{A}^\top\!\boldsymbol{A}$ multiplied by $\boldsymbol{v}^\top\boldsymbol{M_A}\boldsymbol{v}/\boldsymbol{v}^\top\boldsymbol{v}$, where $\boldsymbol{M_A}$ projects orthogonally on to $\mathcal{S}^\perp(\boldsymbol{A})$. **Hint:** Construct a 2×2 matrix $\boldsymbol{C}$ such that the columns of $\boldsymbol{AC}$ are orthonormal, with the first being parallel to $\boldsymbol{P_A v}$.

We have that

$$\boldsymbol{AB} = [\boldsymbol{a}_1 \;\; \boldsymbol{a}_2] \begin{bmatrix} \|\boldsymbol{a}_1\|^{-1} & -\|\boldsymbol{a}_1\|^{-1} \cot\theta \\ 0 & \|\boldsymbol{a}_2\|^{-1} \operatorname{cosec}\theta \end{bmatrix}.$$

The first column of this matrix product is $\boldsymbol{a}_1/\|\boldsymbol{a}_1\|$, of which the norm is 1. The second column is

$$-\frac{\boldsymbol{a}_1}{\|\boldsymbol{a}_1\|}\cot\theta + \frac{\boldsymbol{a}_2}{\|\boldsymbol{a}_2\|}\operatorname{cosec}\theta. \tag{S14.25}$$

The squared norm of this vector is

$$\begin{aligned}\cot^2\theta - 2\cos\theta\cot\theta\operatorname{cosec}\theta + \operatorname{cosec}^2\theta &= \frac{1}{\sin^2\theta}(\cos^2\theta - 2\cos^2\theta + 1) \\ &= \frac{1}{\sin^2\theta}(1 - \cos^2\theta) = 1.\end{aligned}$$

The scalar product of (S14.25) and $\boldsymbol{a}_1/\|\boldsymbol{a}_1\|$ is

$$-\cot\theta + \cos\theta\operatorname{cosec}\theta = 0.$$

Thus the columns of $\boldsymbol{AB}$ are orthonormal.

It follows that $\boldsymbol{B}^\top\!\boldsymbol{A}^\top\!\boldsymbol{AB}$ is a 2×2 identity matrix, of which the determinant is 1. The determinant can also be expressed as the square of the determinant of $\boldsymbol{B}$ times the determinant of $\boldsymbol{A}^\top\!\boldsymbol{A}$. Since $\boldsymbol{B}$ is upper triangular, its determinant is the product of its diagonal elements, that is, $(\|\boldsymbol{a}_1\|\|\boldsymbol{a}_2\|\sin\theta)^{-1}$. Consequently, the determinant of $\boldsymbol{A}^\top\!\boldsymbol{A}$ is equal to $\|\boldsymbol{a}_1\|^2\|\boldsymbol{a}_2\|^2\sin^2\theta$. Since θ is by definition the angle between $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$, we see that $\|\boldsymbol{M}_1\boldsymbol{a}_2\|^2 = \|\boldsymbol{a}_2\|^2 \sin^2\theta$; recall Figure 12.1. This proves the result of the first part of the exercise.

If $\boldsymbol{A}^\top\boldsymbol{v} = \boldsymbol{0}$, the result of the second part is trivial, because $\boldsymbol{M_v A} = \boldsymbol{A}$ and $\boldsymbol{M_A v} = \boldsymbol{v}$. Suppose therefore that $\boldsymbol{P_A v} \neq \boldsymbol{0}$. This vector can therefore be expressed as $c_1\boldsymbol{a}_1 + c_2\boldsymbol{a}_2$ for two scalars c_1 and c_2, not both zero. Let the 2×2 matrix $\boldsymbol{C}$ be given by

$$\boldsymbol{C} \equiv [\boldsymbol{c}_1 \;\; \boldsymbol{c}_2] = \begin{bmatrix} c_1\|\boldsymbol{P_A v}\|^{-1} & d_1 \\ c_2\|\boldsymbol{P_A v}\|^{-1} & d_2 \end{bmatrix},$$

where d_1 and d_2 are chosen so that the second column of $\boldsymbol{AC}$ is orthogonal to the first and has unit norm. It is easy to check that the first column of $\boldsymbol{AC}$, which is $\boldsymbol{Ac}_1$, is $\boldsymbol{P_A v}/\|\boldsymbol{P_A v}\|$, which also has unit norm. The second column, which is $\boldsymbol{Ac}_2$, is orthogonal to $\boldsymbol{v}$, since it is orthogonal both to $\boldsymbol{P_A v}$, by construction, and to $\boldsymbol{M_A v}$, since it belongs to $\mathcal{S}(\boldsymbol{A})$.

The orthogonality of $\boldsymbol{Ac}_2$ and $\boldsymbol{v}$ implies that $\boldsymbol{M_v AC} = [\boldsymbol{M_v Ac}_1 \;\; \boldsymbol{Ac}_2]$, where the two columns on the right-hand side are orthogonal, since $\boldsymbol{c}_1^\top\boldsymbol{A}^\top\boldsymbol{M_v Ac}_2 = \boldsymbol{c}_1^\top\boldsymbol{A}^\top\boldsymbol{Ac}_2 = 0$, because $\boldsymbol{Ac}_1$ and $\boldsymbol{Ac}_2$ are orthogonal by construction. The determinant of $\boldsymbol{C}^\top\boldsymbol{A}^\top\boldsymbol{M_v AC}$ is thus the product of the squared norms of the two columns of $\boldsymbol{M_v AC}$, by the result of the first part of the exercise. The second column has norm 1, and so the determinant is $\|\boldsymbol{M_v Ac}_1\|^2$.

Let ϕ denote the angle between $\boldsymbol{v}$ and $\boldsymbol{P_A v}$. It follows that $\cos^2\phi = \boldsymbol{v}^\top\boldsymbol{P_A v}/\boldsymbol{v}^\top\boldsymbol{v}$ and that $\sin^2\phi = \boldsymbol{v}^\top\boldsymbol{M_A v}/\boldsymbol{v}^\top\boldsymbol{v}$. Because $\boldsymbol{P_A v}$ and $\boldsymbol{Ac}_1$ are parallel, the angle between $\boldsymbol{Ac}_1$ and the vector $\boldsymbol{M_v Ac}_1$, orthogonal to $\boldsymbol{v}$ and in the same plane as $\boldsymbol{v}$ and $\boldsymbol{P_A v}$, is $\pi/2-\phi$. Since $\boldsymbol{Ac}_1$ has unit norm, it follows that $\|\boldsymbol{M_v Ac}_1\| = \|\boldsymbol{Ac}_1\|\cos(\pi/2-\phi) = \sin\phi$.

The determinant of $\boldsymbol{C}^\top\boldsymbol{A}^\top\boldsymbol{M_v AC}$ is therefore equal to $\sin^2\phi$. It is also equal to the square of $|\boldsymbol{C}|$ times the determinant of $\boldsymbol{A}^\top\boldsymbol{M_v A}$. Since the columns of $\boldsymbol{AC}$ are orthonormal, the determinant of $\boldsymbol{C}^\top\boldsymbol{A}^\top\boldsymbol{AC}$ is 1. Thus the determinant of $\boldsymbol{A}^\top\boldsymbol{A}$ is $1/|\boldsymbol{C}|^2$. We therefore have that

$$\frac{|\boldsymbol{A}^\top\boldsymbol{M_v A}|}{|\boldsymbol{A}^\top\boldsymbol{A}|} = \sin^2\phi = \frac{\boldsymbol{v}^\top\boldsymbol{M_A v}}{\boldsymbol{v}^\top\boldsymbol{v}}.$$

This is the result we wished to prove.