Solution to Exercise 14.15

⋆14.15 Show that, if w_t is the standardized random walk (14.01), $\sum_{t=1}^{n} w_t$ is of order $n^{3/2}$ as $n \to \infty$. By use of the definition (14.28) of the Riemann integral, show that

$$\operatorname*{plim}_{n\to\infty} n^{-3/2} \sum_{t=1}^{n} w_t = \int_0^1 W(r)\,dr,$$

and demonstrate that this plim is distributed as $\mathrm{N}(0, {}^1\!/_3)$. **Hint:** Use the results of Exercise 14.4.

Show that the probability limit of the formula (14.20) for the statistic z_c can be written in terms of a standardized Wiener process $W(r)$ as

$$\operatorname*{plim}_{n\to\infty} z_c = \frac{\frac{1}{2}(W^2(1) - 1) - W(1)\int_0^1 W(r)\,dr}{\int_0^1 W^2(r)\,dr - \big(\int_0^1 W(r)\,dr\big)^2}.$$

From equation (14.02), we see that

$$\sum_{t=1}^{n} w_t = \sum_{t=1}^{n}\Big(\sum_{s=1}^{t} \varepsilon_s\Big).$$

Exchanging the order of the two summations gives

$$\sum_{t=1}^{n} w_t = \sum_{s=1}^{n}\sum_{t=s}^{n} \varepsilon_s = \sum_{s=1}^{n}(n-s+1)\varepsilon_s.$$

The easiest way to find the order of this expression is to calculate its variance, which, since the ε_s are IID with variance 1, is

$$\sum_{s=1}^{n}(n-s+1)^2 = \sum_{t=1}^{n} t^2 = \frac{1}{6}n(n+1)(2n+1),$$

where, for the first equality, we make the substitution $t = n - s + 1$, and, for the second, we use one of the results of Exercise 14.4. Thus the variance of $\sum_{t=1}^{n} w_t$ is $O(n^3)$, and it follows that $\sum_{t=1}^{n} w_t$ itself is $O_p(n^{3/2})$.

From the definition of $W(r)$, we have

$$n^{-3/2}\sum_{t=1}^{n} w_t \overset{a}{=} \frac{1}{n}\sum_{t=1}^{n} W\Big(\frac{t}{n}\Big).$$

The right-hand side involves a factor of n^{-1} instead of $n^{-3/2}$ because of the factor of $n^{-1/2}$ in the definition (14.26). Using the definition (14.28) of the Riemann integral, this implies that

$$\operatorname*{plim}_{n\to\infty} n^{-3/2}\sum_{t=1}^{n} w_t = \int_0^1 W(r)\,dr.$$

Since the w_t are normally distributed with mean 0, this equation implies that $\int_0^1 W(r)\,dr$ is normally distributed with mean 0. For the variance, we use the earlier result that

$$\operatorname{Var}\Bigl(\sum_{t=1}^{n} w_t\Bigr) = \tfrac{1}{6}n(n+1)(2n+1).$$

Therefore

$$\lim_{n\to\infty}\operatorname{Var}(n^{-3/2}\sum_{t=1}^{n} w_t) = \lim_{n\to\infty} n^{-3}\;\tfrac{1}{6}n(n+1)(2n+1) = \tfrac{1}{3}.$$

Thus we conclude that

$$\int_0^1 W(r)\,dr \sim \mathrm{N}(0, 1/3),$$

as we were required to show.

According to (14.20), the z_c statistic can be written as

$$z_c = n\frac{\sum_{t=1}^{n}(\boldsymbol{M_\iota w})_{t-1}\varepsilon_t}{\sum_{t=1}^{n}(\boldsymbol{M_\iota w})_{t-1}^2} = \frac{n^{-1}\sum_{t=1}^{n}(\boldsymbol{M_\iota w})_{t-1}\varepsilon_t}{n^{-2}\sum_{t=1}^{n}(\boldsymbol{M_\iota w})_{t-1}^2}. \tag{S14.14}$$

The numerator of the rightmost expression here is

$$\begin{aligned}\frac{1}{n}\sum_{t=1}^{n}(\boldsymbol{M_\iota w})_{t-1}\varepsilon_t &= \frac{1}{n}\sum_{t=1}^{n} w_{t-1}\varepsilon_t - \bar{w}\sum_{t=1}^{n}\varepsilon_t\\ &= \frac{1}{n}\sum_{t=1}^{n} w_{t-1}\varepsilon_t - \Bigl(n^{-3/2}\sum_{t=1}^{n} w_{t-1}\Bigr)\Bigl(n^{-1/2}\sum_{t=1}^{n}\varepsilon_t\Bigr),\end{aligned}$$

where $\bar{w}$ is the mean of the w_t from 0 to $n-1$. From equation (14.27), the plim of the first term in the second line is $\frac{1}{2}\bigl(W^2(1)-1\bigr)$. From the result proved in the first part of this exercise, the plim of the first factor in the second term is $\int_0^1 W(r)\,dr$. Finally, since the second factor in the second term is just $n^{-1/2}w_n$, its plim is $W(1)$. Thus we conclude that the plim of the numerator of the z_c statistic is

$$\tfrac{1}{2}\bigl(W^2(1)-1\bigr) - W(1)\int_0^1 W(r)\,dr, \tag{S14.15}$$

which is what we were required to show.

Similarly, the denominator of the rightmost expression in (S14.14) is

$$n^{-2}\sum_{t=1}^{n}(\boldsymbol{M_\iota w})_{t-1}^2 = n^{-2}\sum_{t=1}^{n} w_{t-1}^2 - \biggl(n^{-3/2}\sum_{t=1}^{n} w_{t-1}\biggr)^2.$$

From equation (14.29), we conclude that the first term here has a plim of $\int_0^1 W^2(r)\,dr$. From the result proved in the first part of this exercise, the second term has a plim of $-\bigl(\int_0^1 W(r)\,dr\bigr)^2$. Thus we conclude that the plim of the denominator of the z_c statistic is

$$\int_0^1 W^2(r)\,dr - \biggl(\int_0^1 W(r)\,dr\biggr)^2, \tag{S14.16}$$

which is what we were required to show. Combining (S14.15) with (S14.16) gives us the expression for the plim of z_c that was given in the question.