Solution to Exercise 14.14

★**14.14** By using arguments similar to those leading to the result (14.30) for the z_{nc} statistic, demonstrate the result (14.31) for the τ_{nc} statistic.

The test statistic τ_{nc} is the t statistic for $\beta - 1 = 0$ in regression (14.15). Using the result of Exercise 4.9, this is $(n-1)^{1/2}$ times the cotangent of the angle between the vector with typical element Δy_t and that with typical element y_{t-1}. Under the null hypothesis, $y_t = \sigma w_t$, where w_t is a standardized random walk. The angle is thus the same as the angle between the vectors with typical elements $\Delta w_t = \varepsilon_t$ and w_{t-1}.

For any two nonzero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, the cosine of the angle between them is $\boldsymbol{a}^\top\boldsymbol{b}/\big(\|\boldsymbol{a}\|\,\|\boldsymbol{b}\|\big)$. For any angle $\theta \in [0, \pi]$, $\cot\theta = \cos\theta/(1-\cos^2\theta)^{1/2}$, and so the cotangent of the angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ is

$$\boldsymbol{a}^\top\boldsymbol{b}\big(\|\boldsymbol{a}\|\,\|\boldsymbol{b}\| - (\boldsymbol{a}^\top\boldsymbol{b})^2\big)^{-1/2}.$$

Thus the τ_{nc} statistic can be written as

$$\frac{(n-1)^{1/2}\sum_{t=2}^n w_{t-1}\varepsilon_t}{\Big(\sum_{t=2}^n w_{t-1}^2 \sum_{t=1}^n \varepsilon_t^2 - \big(\sum_{t=2}^n w_{t-1}\varepsilon_t\big)^2\Big)^{1/2}}. \tag{S14.13}$$

The plims of most of the sums above were worked out in the text for z_{nc}. The only one not yet considered is the sum of the ε_t^2. By a law of large numbers,

$$\operatorname*{plim}_{n\to\infty} \frac{1}{n}\sum_{t=1}^n \varepsilon_t^2 = \mathrm{E}(\varepsilon_t^2) = 1.$$

Thus the first term in the expression of which we compute the square root in (S14.13) is $O_p(n^3)$, that is, the order of $\sum w_{t-1}^2$ plus that of $\sum \varepsilon_t^2$, whereas the second term is $O_p(n^2)$, since $\sum w_{t-1}\varepsilon_t = O_p(n)$. In computing the plim of (S14.13), therefore, we can ignore the second term in the denominator. From (14.27) and (14.29), it is then easy to check that

$$\operatorname*{plim}_{n\to\infty} \tau_{nc} = \frac{\frac{1}{2}\big(W^2(1) - 1\big)}{\big(\int_0^1 W^2(r)\,dr\big)^{1/2}},$$

which is equation (14.31).