Solution to Exercise 14.12

⋆14.12 Show that

$$\sum_{t=1}^{n} w_t^2 = \sum_{t=1}^{n}(n-t+1)\varepsilon_t^2 + 2\sum_{t=2}^{n}\sum_{s=1}^{t-1}(n-t+1)\varepsilon_t\varepsilon_s, \tag{S14.09}$$

where w_t is the standardized random walk (14.02). Then demonstrate that a typical term from the first summation is uncorrelated with every other term from the first summation and with every term from the second (double) summation. Also demonstrate that every term from the double summation is uncorrelated with every other such term.

Let the fourth moment of the white-noise process ε_t be m_4. Show that the variance of $\sum_{t=1}^{n} w_t^2$ is then equal to

$$\frac{m_4}{6}n(n+1)(2n+1) + \tfrac{1}{3}n^2(n^2-1),$$

which is of order n^4 as $n \to \infty$. **Hint:** Use the results of Exercise 14.4.

From equation (14.02), we have that

$$\sum_{t=1}^{n} w_t^2 = \sum_{t=1}^{n}\Bigg(\sum_{s=1}^{t}\varepsilon_s\Bigg)^2 = \sum_{t=1}^{n}\sum_{s_1=1}^{t}\sum_{s_2=1}^{t}\varepsilon_{s_1}\varepsilon_{s_2}. \tag{S14.10}$$

Consider first the terms in the last expression above for which $s_1 = s_2 = s$. The sum of these terms is

$$\sum_{t=1}^{n}\sum_{s=1}^{t}\varepsilon_s^2 = \sum_{s=1}^{n}\sum_{t=s}^{n}\varepsilon_s^2 = \sum_{s=1}^{n}(n-s+1)\varepsilon_s^2.$$

In the first step, we interchange the order of the sums over t and s, and, in the second, we note that the summands do not depend on t and that the sum over t has $n-s+1$ terms for given s. Replacing the dummy index s by t shows that this sum is equal to the first term on the right-hand side of equation (S14.09).

When we consider the terms of (S14.10) for which $s_1 \neq s_2$, we see that, since $\varepsilon_{s_1}\varepsilon_{s_2}$ is symmetric with respect to s_1 and s_2, we can group these terms as follows:

$$2\sum_{t=1}^{n}\sum_{s_1=2}^{t}\sum_{s_2=1}^{s_1-1}\varepsilon_{s_1}\varepsilon_{s_2}.$$

Exchanging the order of the sums over t and s_1, we can rewrite this as

$$2\sum_{s_1=2}^{n}\sum_{t=s_1}^{n}\sum_{s_2=1}^{s_1-1}\varepsilon_{s_1}\varepsilon_{s_2} = 2\sum_{s_1=2}^{n}\sum_{s_2=1}^{s_1-1}(n-s_1+1)\varepsilon_{s_1}\varepsilon_{s_2}.$$

Except for the dummy indices, the right-hand side above is just the second term on the right-hand side of (S14.09).

That a typical term from the first summation is uncorrelated with every other term from the first summation is easy to see. By assumption, the ε_t are independent. Therefore, ε_t^2 must be independent of, and therefore uncorrelated with, ε_s^2 for all $s \neq t$.

That a typical term from the first summation is uncorrelated with every term from the second summation is also not hard to see. Ignoring the constant factors, these two terms are

$$\varepsilon_t^2 \quad \text{and} \quad \varepsilon_\tau \varepsilon_s, \quad s = 1, \ldots, \tau - 1.$$

When $t \neq \tau$ and $t \neq s$, the correlation is obviously 0, since each term is a function of independent random variables. When $t = \tau$, what we have is the correlation between ε_t^2 and $\varepsilon_t \varepsilon_s$. The fact that ε_s is independent of ε_t implies that this correlation must be 0. Similarly, if $t = s$, the correlation of ε_s^2 and $\varepsilon_\tau \varepsilon_s$ is zero since $\tau \neq s$.

That a typical term from the double summation is uncorrelated with every other such term follows by similar arguments. In order for the expectation of the product $\varepsilon_{s_1}\varepsilon_{s_2}\varepsilon_{s_3}\varepsilon_{s_4}$ to be nonzero, either the indices must all be equal or else each index must be equal to one other index. But no such product can be constructed by multiplying together two distinct terms of the double summation.

Since each term on the right-hand side of equation (S14.09) is uncorrelated with every other term, the variance of the entire expression is the sum of the variances of the individual terms. Term t in the first, single, sum has a variance of $m_4(n - t + 1)^2$, and so the sum of the variances of the terms in the single sum is

$$m_4 \sum_{t=1}^{n} (n - t + 1)^2 = m_4 \sum_{t=1}^{n} t^2 = \frac{m_4}{6} n(n + 1)(2n + 1).$$

In the first step, we reorder the terms of the sum, replacing $n - t + 1$ by t, and, in the second step, we use one of the results of Exercise 14.4. If the ε_t followed the standard normal distribution, m_4 would be 3.

The variance of the term indexed by t and s in the double sum is $4(n-t+1)^2$, where the factor of 4 is the square of the factor of 2 that multiplies the double sum. Note that this variance does not depend on s. The sum of the variances of all the terms in the double sum that correspond to the index t is therefore just $4(t-1)(n-t+1)^2$. Summing this over t from $t = 2$ to $t = n$ is equivalent to summing the terms $4t(n - t)^2$ from $t = 1$ to $t = n - 1$. Making use once more of the results of Exercise 14.4, we obtain

$$
\begin{aligned}
\sum_{t=1}^{n-1} 4t(n-t)^2 &= 4n^2\sum_{t=1}^{n-1} t - 8n\sum_{t=1}^{n-1} t^2 + 4\sum_{t=1}^{n-1} t^3 \\
&= 2n^3(n-1) - \tfrac{4}{3}n^2(n-1)(2n-1) + n^2(n-1)^2 \\
&= \tfrac{1}{3}n^2(n-1)(6n-8n+4+3n-3) \\
&= \tfrac{1}{3}n^2(n-1)(n+1) = \tfrac{1}{3}n^2(n^2-1).
\end{aligned}
$$

Adding the sum of the variances of the terms of the single sum to the sum of the variances of the terms of the double sum gives the desired result.