Solution to Exercise 14.11

⋆14.11 Consider the model with typical DGP

$$y_t = \sum_{i=0}^{p} \beta_i t^i + y_{t-1} + \sigma\varepsilon_t, \quad \varepsilon_t \sim \mathrm{IID}(0,1). \tag{14.79}$$

Show that the z and τ statistics from the testing regression

$$\Delta y_t = \sum_{i=0}^{p+1} \gamma_i t^i + (\beta - 1)y_{t-1} + e_t$$

are pivotal when the DGP is (14.79) and the distribution of the white-noise process ε_t is known.

We saw in Exercise 14.5 that the DGP

$$y_t = \beta_0 + \beta_1 t + \beta_2 t^2 + y_{t-1} + u_t, \quad u_t \sim \mathrm{IID}(0, \sigma^2),$$

is equivalent to

$$y_t = y_0 + \beta_0 t + \beta_1 \frac{1}{2} t(t+1) + \beta_2 \frac{1}{6} t(t+1)(2t+1) + \sigma w_t, \tag{S14.07}$$

where w_t is a standardized random walk, and where the right-hand side can be written as

$$y_0 + \sum_{i=1}^{3} \gamma_i t^i + \sigma w_t$$

with a suitable definition of the γ_i, $i = 1, 2, 3$. The result of Exercise 14.5 was extended in such a way that we saw that the DGP (14.79) is equivalent to

$$y_t = y_0 + \sum_{i=1}^{p+1} \gamma_i t^i + \sigma w_t, \tag{S14.08}$$

again with an appropriate definition of the γ_i. An implication of the equivalence of (14.79) and (S14.08) is that the z and τ statistics generally depend on the γ_i. Expression (14.17) shows that this is the case for z whenever $p = 0$ and the testing regression does not include a constant term.

Now suppose that we include t^i, for $i = 0, \ldots, p+1$, in the DF test regression. Let $\boldsymbol{T}$ denote the matrix of these deterministic regressors. It is an $n \times (p+2)$ matrix of which the first column is a vector of 1s, the second column has typical element t, the third column has typical element t^2, and so on. Let

$\boldsymbol{M_T}$ denote the matrix that projects orthogonally on to $\mathcal{S}^\perp(\boldsymbol{T})$. By the FWL Theorem, the DF test regression is equivalent to the FWL regression

$$\boldsymbol{M_T}\Delta\boldsymbol{y} = (\beta - 1)\boldsymbol{M_T}\boldsymbol{y}_{-1} + \boldsymbol{e},$$

where the notation should be obvious. Therefore, we can write the Dickey-Fuller z statistic as

$$z = n\,\frac{\boldsymbol{y}_{-1}^\top\boldsymbol{M_T}\Delta\boldsymbol{y}}{\boldsymbol{y}_{-1}^\top\boldsymbol{M_T}\boldsymbol{y}_{-1}}.$$

Since the data are generated by equation (S14.08), $\boldsymbol{M_T}$ annihilates the constant term y_0 and all the trend terms, and we see that $\boldsymbol{M_T}\boldsymbol{y}_{-1}$ is equal to $\sigma\boldsymbol{M_T}\boldsymbol{w}_{-1}$, where $\boldsymbol{w}_{-1}$ is a vector with typical element w_{t-1}. Similarly, $\boldsymbol{M_T}\Delta\boldsymbol{y}$ is equal to $\sigma\boldsymbol{M_T}(\boldsymbol{w} - \boldsymbol{w}_{-1})$. Thus

$$z = n\,\frac{\sigma^2\boldsymbol{w}_{-1}^\top\boldsymbol{M_T}(\boldsymbol{w} - \boldsymbol{w}_{-1})}{\sigma^2\boldsymbol{w}_{-1}^\top\boldsymbol{M_T}\boldsymbol{w}_{-1}} = n\,\frac{\boldsymbol{w}_{-1}^\top\boldsymbol{M_T}(\boldsymbol{w} - \boldsymbol{w}_{-1})}{\boldsymbol{w}_{-1}^\top\boldsymbol{M_T}\boldsymbol{w}_{-1}}.$$

The rightmost expression here is evidently pivotal if the distribution of the error terms is known, since it depends only on the deterministic variables in $\boldsymbol{T}$ and the standardized random walk process.

Similarly, we can write the Dickey-Fuller τ statistic as

$$\tau = \left(\frac{\Delta\boldsymbol{y}^\top\boldsymbol{M_T}\boldsymbol{M}_{\boldsymbol{M_T}\boldsymbol{y}_{-1}}\boldsymbol{M_T}\Delta\boldsymbol{y}}{n-p-3}\right)^{-1/2}\frac{\boldsymbol{y}_{-1}^\top\boldsymbol{M_T}\Delta\boldsymbol{y}}{(\boldsymbol{y}_{-1}^\top\boldsymbol{M_T}\boldsymbol{y}_{-1})^{1/2}}.$$

Under the DGP (S14.08), we can once again replace $\boldsymbol{M_T}\boldsymbol{y}_{-1}$ by $\sigma\boldsymbol{M_T}\boldsymbol{w}_{-1}$ and $\boldsymbol{M_T}\Delta\boldsymbol{y}$ by $\sigma\boldsymbol{M_T}(\boldsymbol{w} - \boldsymbol{w}_{-1})$. Thus the test statistic depends only on $\boldsymbol{T}$, $\boldsymbol{w}$, and σ. But the various powers of σ cancel, and we are left with

$$\left(\frac{(\boldsymbol{w} - \boldsymbol{w}_{-1})^\top\boldsymbol{M_T}\boldsymbol{M}_{\boldsymbol{M_T}\boldsymbol{w}_{-1}}\boldsymbol{M_T}(\boldsymbol{w} - \boldsymbol{w}_{-1})}{n-p-3}\right)^{-1/2}\frac{\boldsymbol{w}_{-1}^\top\boldsymbol{M_T}(\boldsymbol{w} - \boldsymbol{w}_{-1})}{(\boldsymbol{w}_{-1}^\top\boldsymbol{M_T}\boldsymbol{w}_{-1})^{1/2}},$$

which again is evidently pivotal when the distribution of the ε_t is known.