Solution to Exercise 13.6

⋆13.6 Show that the method illustrated in Section 13.2 for obtaining the autocovariances of an ARMA$(1,1)$ process can be extended to the ARMA(p,q) case. Since explicit formulas are hard to obtain for general p and q, it is enough to indicate a recursive method for obtaining the solution.

An ARMA(p,q) process with no constant term can be written as

$$u_t = \sum_{i=1}^{p} \rho_i u_{t-i} + \varepsilon_t + \sum_{j=1}^{q} \alpha_j \varepsilon_{t-j}. \tag{S13.12}$$

If we multiply this equation by u_t and take expectations, we get

$$v_0 = \sum_{i=1}^{p} \rho_i v_i + \sigma_\varepsilon^2 + \sum_{j=1}^{q} \alpha_j w_j.$$

Similarly, if we multiply equation (S13.12) by u_{t-1} and take expectations, we obtain the equation

$$v_1 = \sum_{i=1}^{p} \rho_i v_{i-1} + \sum_{j=1}^{q} \alpha_j w_{j-1},$$

and if we multiply it by u_{t-2} and take expectations, we obtain the equation

$$v_2 = \sum_{i=1}^{p} \rho_i v_{|i-2|} + \sum_{j=2}^{q} \alpha_j w_{j-2}.$$

Notice that, in the first summation, we sum over all $i = 1, \ldots, p$ while, in the second, we sum only over $j = 2, \ldots, q$. The absolute values appear in the first summation because $v_{|i-2|}$ is the expectation of $u_{t-2}u_{t-i}$. We can continue in this fashion, multiplying equation (S13.12) by u_{t-l} for $l \geq 0$ and taking expectations, to obtain the general result that

$$v_l = \sum_{i=1}^{p} \rho_i v_{|i-l|} + \sum_{j=l}^{q} \alpha_j w_{j-l}, \tag{S13.13}$$

where it is understood that, if $l > q$, the second sum is zero. Note that, for $l \leq p$, only the v_i for $i = 0, \ldots, p$ appear in this equation.

In order to find the w_j, we need to multiply equation (S13.12) by ε_{t-l} and take expectations. For $l = 0$, we get

$$w_0 = \sigma_\varepsilon^2.$$

For $l = 1$, we get

$$w_1 = \rho_1 w_0 + \alpha_1 \sigma_\varepsilon^2.$$

Both of these equations are exactly the same as we got in the ARMA$(1,1)$ case. For $l = 2$, we get

$$w_2 = \rho_1 w_1 + \rho_2 w_0 + \alpha_2 \sigma_\varepsilon^2.$$

Thus, it is clear that, for arbitrary $l \geq 1$, the w_l are determined by the recursion

$$w_l = \sum_{i=1}^{l} \rho_i w_{l-i} + \alpha_l \sigma_\varepsilon^2,$$

where $\alpha_l = 0$ for $l > q$. Starting with $w_0 = \sigma_\varepsilon^2$, we can solve this recursion for the w_l. The solutions can then be plugged into the first $p+1$ of equations (S13.13) and those equations solved to obtain v_0 through v_p. Once we have v_0 through v_p and w_0 through w_q, equations (S13.13) can then be used to generate the autocovariances recursively for lags greater than p.