Solution to Exercise 12.8

⋆12.8 Suppose that m independent random variables, z_i, each of which is distributed as $\mathrm{N}(0, 1)$, are grouped into an m–vector $\boldsymbol{z}$. Let $\boldsymbol{x} = \boldsymbol{\mu} + \boldsymbol{A}\boldsymbol{z}$, where $\boldsymbol{\mu}$ is an m–vector and $\boldsymbol{A}$ is a nonsingular $m \times m$ matrix, and let $\boldsymbol{\Omega} \equiv \boldsymbol{A}\boldsymbol{A}^\top$. Show that the mean of the vector $\boldsymbol{x}$ is $\boldsymbol{\mu}$ and its covariance matrix is $\boldsymbol{\Omega}$. Then show that the density of $\boldsymbol{x}$ is

$$(2\pi)^{-m/2}|\boldsymbol{\Omega}|^{-1/2}\exp\bigl(-\tfrac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^\top\boldsymbol{\Omega}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\bigr). \tag{12.122}$$

This extends the result of Exercise 4.5 for the bivariate normal density to the multivariate normal density. **Hints:** Remember that the joint density of m independent random variables is equal to the product of their densities, and use the result (12.29).

The first result is trivial to prove. Clearly,

$$\mathrm{E}(\boldsymbol{x}) = \mathrm{E}(\boldsymbol{\mu} + \boldsymbol{A}\boldsymbol{z}) = \boldsymbol{\mu} + \boldsymbol{A}\mathrm{E}(\boldsymbol{z}) = \boldsymbol{\mu}.$$

For the second result, we have

$$\begin{aligned}\mathrm{E}\bigl((\boldsymbol{x}-\boldsymbol{\mu})(\boldsymbol{x}-\boldsymbol{\mu})^\top\bigr) &= \mathrm{E}(\boldsymbol{A}\boldsymbol{z}\boldsymbol{z}^\top\boldsymbol{A}^\top) = \boldsymbol{A}\mathrm{E}(\boldsymbol{z}\boldsymbol{z}^\top)\boldsymbol{A}^\top\\ &= \boldsymbol{A}\mathbf{I}\boldsymbol{A}^\top = \boldsymbol{A}\boldsymbol{A}^\top = \boldsymbol{\Omega},\end{aligned}$$

as we were required to show.

The third result requires a little bit more work. The density of each of the z_i is the standard normal density,

$$f(z_i) = (2\pi)^{-1/2}\exp\bigl(-\tfrac{1}{2}z_i^2\bigr).$$

Since the z_i are independent, the joint density of all of them is just the product of m of these densities, which is

$$(2\pi)^{-m/2}\exp\Bigl(-\frac{1}{2}\sum_{i=1}^{m}z_i^2\Bigr) = (2\pi)^{-m/2}\exp\bigl(-\tfrac{1}{2}\boldsymbol{z}^\top\boldsymbol{z}\bigr). \tag{S12.10}$$

Next, we use the result (12.29), which requires that we replace the vector $\boldsymbol{z}$ in the expression on the right-hand side of equation (S12.10) by

$$\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{A}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})$$

and then multiply by the determinant of the Jacobian of the transformation, which is

$$|\boldsymbol{A}^{-1}| = |\boldsymbol{A}|^{-1} = |\boldsymbol{\Omega}|^{-1/2}.$$

We do not need to take the absolute value of the determinant in this case, because $\boldsymbol{\Omega}$, and hence also $\boldsymbol{A}$ and its inverse, are positive definite matrices. Thus the result of using (12.29) is

$$(2\pi)^{-m/2}|\boldsymbol{\Omega}|^{-1/2}\exp\bigl(-\tfrac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^{\top}(\boldsymbol{A}^{-1})^{\top}\boldsymbol{A}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\bigr).$$

This can be rewritten as (12.122) by using the fact that

$$(\boldsymbol{A}^{-1})^{\top}\boldsymbol{A}^{-1} = (\boldsymbol{A}\boldsymbol{A}^{\top})^{-1} = \boldsymbol{\Omega}^{-1}.$$

Thus we conclude that the joint density of the vector $\boldsymbol{x}$ is expression (12.122), as we were required to show.