Solution to Exercise 12.22

★12.22 It was shown in Section 12.4 that the rank condition for the asymptotic identification of equation (12.72) is that the $(l - k_{11}) \times k_{21}$ matrix $\boldsymbol{\Pi}_{21}$ of the unrestricted reduced form (12.73) should have full column rank. Show that, in terms of the structural parameters, $\boldsymbol{\Pi}_{21}$ is equal to $\boldsymbol{B}_{22}\boldsymbol{\Gamma}^{11}$. Then consider the matrix

$$\begin{bmatrix} \boldsymbol{\Gamma}_{22} \\ \boldsymbol{B}_{22} \end{bmatrix}, \tag{12.125}$$

and show, by postmultiplying it by the nonsingular matrix $[\boldsymbol{\Gamma}^{11} \;\; \boldsymbol{\Gamma}^{12}]$, that it is of full column rank $g - 1$ if and only if $\boldsymbol{B}_{22}\boldsymbol{\Gamma}^{11}$ is of full column rank. Conclude that the rank condition for the asymptotic identification of (12.72) is that (12.125) should have full column rank.

Imposing the overidentifying restrictions on the matrix $\boldsymbol{\Pi}$ of coefficients of the reduced form gives us that $\boldsymbol{\Pi} = \boldsymbol{B}\boldsymbol{\Gamma}^{-1}$, or, explicitly,

$$\boldsymbol{\Pi} = \begin{bmatrix} \boldsymbol{\beta}_{11} & \boldsymbol{B}_{12} \\ \mathbf{0} & \boldsymbol{B}_{22} \end{bmatrix} \begin{bmatrix} \gamma^{00} & \boldsymbol{\Gamma}^{01} & \boldsymbol{\Gamma}^{02} \\ \boldsymbol{\gamma}^{10} & \boldsymbol{\Gamma}^{11} & \boldsymbol{\Gamma}^{12} \end{bmatrix}.$$

The matrix

$$\begin{bmatrix} \boldsymbol{\pi}_{11} & \boldsymbol{\Pi}_{11} \\ \boldsymbol{\pi}_{21} & \boldsymbol{\Pi}_{21} \end{bmatrix}$$

that occurs in equation (12.73) is the matrix $\boldsymbol{\Pi}$ without its last $g - k_{21} - 1$ columns, that is, the columns that correspond to the endogenous variables excluded from equation 1. Thus

$$\begin{bmatrix} \boldsymbol{\pi}_{11} & \boldsymbol{\Pi}_{11} \\ \boldsymbol{\pi}_{21} & \boldsymbol{\Pi}_{21} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\beta}_{11} & \boldsymbol{B}_{12} \\ \mathbf{0} & \boldsymbol{B}_{22} \end{bmatrix} \begin{bmatrix} \gamma^{00} & \boldsymbol{\Gamma}^{01} \\ \boldsymbol{\gamma}^{10} & \boldsymbol{\Gamma}^{11} \end{bmatrix}.$$

From this, it is clear that

$$\boldsymbol{\Pi}_{21} = \mathbf{0}\boldsymbol{\Gamma}^{01} + \boldsymbol{B}_{22}\boldsymbol{\Gamma}^{11} = \boldsymbol{B}_{22}\boldsymbol{\Gamma}^{11},$$

as we were asked to show.

Next, we postmultiply (12.125) by $[\boldsymbol{\Gamma}^{11} \;\; \boldsymbol{\Gamma}^{12}]$. The result is

$$\begin{bmatrix} \boldsymbol{\Gamma}_{22} \\ \boldsymbol{B}_{22} \end{bmatrix} [\boldsymbol{\Gamma}^{11} \;\; \boldsymbol{\Gamma}^{12}] = \begin{bmatrix} \boldsymbol{\Gamma}_{22}\boldsymbol{\Gamma}^{11} & \boldsymbol{\Gamma}_{22}\boldsymbol{\Gamma}^{12} \\ \boldsymbol{B}_{22}\boldsymbol{\Gamma}^{11} & \boldsymbol{B}_{22}\boldsymbol{\Gamma}^{12} \end{bmatrix}. \tag{S12.30}$$

We saw in the preceding question that the upper leftmost block here is a zero matrix and the upper rightmost block is an identity matrix. Thus the right-hand side of equation (S12.30) becomes

$$\begin{bmatrix} \mathbf{O} & \mathbf{I} \\ \boldsymbol{B}_{22}\boldsymbol{\Gamma}^{11} & \boldsymbol{B}_{22}\boldsymbol{\Gamma}^{12} \end{bmatrix}. \tag{S12.31}$$

Now recall that $\boldsymbol{\Pi}_{21} = \boldsymbol{B}_{22}\boldsymbol{\Gamma}^{11}$ and that the rank condition for identification states that this matrix must have full rank. From the result of Exercise 12.20, we conclude that (S12.31) has full rank if and only if the rank condition is satisfied. But we saw in the previous exercise that $[\boldsymbol{\Gamma}^{11} \;\; \boldsymbol{\Gamma}^{12}]$ has full rank. Therefore, from (S12.30), the matrix (S12.31) has full rank if and only if the matrix (12.125) does. Thus the rank condition for identification is equivalent to the condition that the matrix (12.125) has full column rank.