Solution to Exercise 12.21

★12.21 Consider equation (12.72), the first structural equation of the linear simultaneous system (12.68), with the variables ordered as described in the discussion of the asymptotic identification of this equation. Let the matrices $\boldsymbol{\Gamma}$ and $\boldsymbol{B}$ of the full system (12.68) be partitioned as follows:

$$\boldsymbol{B} = \begin{bmatrix} \boldsymbol{\beta}_{11} & \boldsymbol{B}_{12} \\ \mathbf{0} & \boldsymbol{B}_{22} \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Gamma} = \begin{bmatrix} 1 & \boldsymbol{\Gamma}_{02} \\ -\boldsymbol{\beta}_{21} & \boldsymbol{\Gamma}_{12} \\ \mathbf{0} & \boldsymbol{\Gamma}_{22} \end{bmatrix},$$

where $\boldsymbol{\beta}_{11}$ is a k_{11}–vector, $\boldsymbol{B}_{12}$ and $\boldsymbol{B}_{22}$ are, respectively, $k_{11} \times (g-1)$ and $(l-k_{11}) \times (g-1)$ matrices, $\boldsymbol{\beta}_{21}$ is a k_{21}–vector, and $\boldsymbol{\Gamma}_{02}$, $\boldsymbol{\Gamma}_{12}$, and $\boldsymbol{\Gamma}_{22}$ are, respectively, $1 \times (g-1)$, $k_{21} \times (g-1)$, and $(g-k_{21}-1) \times (g-1)$ matrices. Check that the restrictions imposed in this partitioning correspond correctly to the structure of (12.72).

Let $\boldsymbol{\Gamma}^{-1}$ be partitioned as

$$\boldsymbol{\Gamma}^{-1} = \begin{bmatrix} \gamma^{00} & \boldsymbol{\Gamma}^{01} & \boldsymbol{\Gamma}^{02} \\ \gamma^{10} & \boldsymbol{\Gamma}^{11} & \boldsymbol{\Gamma}^{12} \end{bmatrix},$$

where the rows of $\boldsymbol{\Gamma}^{-1}$ are partitioned in the same pattern as the columns of $\boldsymbol{\Gamma}$, and vice versa. Show that $\boldsymbol{\Gamma}_{22}\boldsymbol{\Gamma}^{12}$ is an identity matrix, and that $\boldsymbol{\Gamma}_{22}\boldsymbol{\Gamma}^{11}$ is a zero matrix, and specify the dimensions of these matrices. Show also that the matrix $[\boldsymbol{\Gamma}^{11} \;\; \boldsymbol{\Gamma}^{12}]$ is square and nonsingular.

With the matrices $\boldsymbol{B}$ and $\boldsymbol{\Gamma}$ partitioned as above, the system (12.68) can be written as

$$[\boldsymbol{y} \;\; \boldsymbol{Y}_1 \;\; \boldsymbol{Y}_2] \begin{bmatrix} 1 & \boldsymbol{\Gamma}_{02} \\ -\boldsymbol{\beta}_{21} & \boldsymbol{\Gamma}_{12} \\ \mathbf{0} & \boldsymbol{\Gamma}_{22} \end{bmatrix} = [\boldsymbol{Z}_1 \;\; \boldsymbol{W}_1] \begin{bmatrix} \boldsymbol{\beta}_{11} & \boldsymbol{B}_{12} \\ \mathbf{0} & \boldsymbol{B}_{22} \end{bmatrix} + [\boldsymbol{u}_1 \;\; \boldsymbol{U}_1],$$

where the $n \times (g-k_{21}-1)$ matrix $\boldsymbol{Y}_2$ contains the endogenous variables excluded from the first equation, and the $n \times (g-1)$ matrix $\boldsymbol{U}_1$ consists of the vectors $\boldsymbol{u}_2$ through $\boldsymbol{u}_g$. The first column of this system is

$$\boldsymbol{y} - \boldsymbol{Y}_1\boldsymbol{\beta}_{21} = \boldsymbol{Z}_1\boldsymbol{\beta}_{11} + \boldsymbol{u}_1,$$

which is simply equation (12.72) with $\boldsymbol{Y}_1\boldsymbol{\beta}_{21}$ moved over to the left-hand side. The remaining columns are

$$\boldsymbol{y}\boldsymbol{\Gamma}_{02} + \boldsymbol{Y}_1\boldsymbol{\Gamma}_{12} + \boldsymbol{Y}_2\boldsymbol{\Gamma}_{22} = \boldsymbol{Z}_1\boldsymbol{B}_{12} + \boldsymbol{W}_1\boldsymbol{B}_{22} + \boldsymbol{U}_1.$$

Both sides of this equation have n rows and $g-1$ columns.

By definition, $\boldsymbol{\Gamma}\boldsymbol{\Gamma}^{-1}$ is an identity matrix. Therefore

$$\begin{bmatrix} 1 & \boldsymbol{\Gamma}_{02} \\ -\boldsymbol{\beta}_{21} & \boldsymbol{\Gamma}_{12} \\ \mathbf{0} & \boldsymbol{\Gamma}_{22} \end{bmatrix} \begin{bmatrix} \gamma^{00} & \boldsymbol{\Gamma}^{01} & \boldsymbol{\Gamma}^{02} \\ \boldsymbol{\gamma}^{10} & \boldsymbol{\Gamma}^{11} & \boldsymbol{\Gamma}^{12} \end{bmatrix} = \mathbf{I}. \tag{S12.29}$$

But the lowest rightmost block of this matrix is $\boldsymbol{\Gamma}_{22}\boldsymbol{\Gamma}^{12}$, and this must be an identity matrix of dimension $g - k_{21} - 1$ if equation (S12.29) is to hold.

Similarly, the matrix $\boldsymbol{\Gamma}_{22}\boldsymbol{\Gamma}^{11}$ must be a zero matrix if equation (S12.29) is to hold. Its dimensions are $(g - k_{21} - 1) \times k_{21}$. The matrix $[\boldsymbol{\Gamma}^{11} \;\; \boldsymbol{\Gamma}^{12}]$ has dimensions $(g - 1) \times (g - 1)$. It is nonsingular, because, as can readily be checked from the implications of (S12.29), its inverse is the matrix

$$\begin{bmatrix} \boldsymbol{\Gamma}_{12} + \boldsymbol{\beta}_{21}\boldsymbol{\Gamma}_{02} \\ \boldsymbol{\Gamma}_{22} \end{bmatrix}.$$