Solution to Exercise 12.2

⋆12.2 Prove the first result of equations (12.08) for an arbitrary $p \times q$ matrix $\boldsymbol{A}$ and an arbitrary $r \times s$ matrix $\boldsymbol{B}$. Prove the second result for $\boldsymbol{A}$ and $\boldsymbol{B}$ as above, and for $\boldsymbol{C}$ and $\boldsymbol{D}$ arbitrary $q \times t$ and $s \times u$ matrices, respectively. Prove the third result in (12.08) for an arbitrary nonsingular $p \times p$ matrix $\boldsymbol{A}$ and nonsingular $r \times r$ matrix $\boldsymbol{B}$.

Give details of the interchanges of rows and columns needed to convert $\boldsymbol{A} \otimes \boldsymbol{B}$ into $\boldsymbol{B} \otimes \boldsymbol{A}$, where $\boldsymbol{A}$ is $p \times q$ and $\boldsymbol{B}$ is $r \times s$.

The first result to be shown is that

$$(\boldsymbol{A} \otimes \boldsymbol{B})^\top = \boldsymbol{A}^\top \otimes \boldsymbol{B}^\top,$$

where $\boldsymbol{A}$ is $p \times q$, and $\boldsymbol{B}$ is $r \times s$. Each element of $\boldsymbol{A} \otimes \boldsymbol{B}$ is of the form $a_{ij} b_{kl}$, and this particular element is in row $(i-1)r+k$ of the Kronecker product, and in column $(j-1)s+l$. This element is thus the element of $(\boldsymbol{A} \otimes \boldsymbol{B})^\top$ in row $(j-1)s+1$ and column $(i-1)r+k$. Now $a_{ij} = \boldsymbol{A}^\top_{ji}$ and $b_{kl} = \boldsymbol{B}^\top_{lk}$. Thus $a_{ij} b_{lk}$ is in row $(j-1)s+l$ and column $(i-1)r+k$ of $\boldsymbol{A}^\top \otimes \boldsymbol{B}^\top$. This shows that $(\boldsymbol{A} \otimes \boldsymbol{B})^\top$ and $\boldsymbol{A}^\top \otimes \boldsymbol{B}^\top$ coincide element by element, and therefore as complete matrices.

The second result to be shown is that

$$(\boldsymbol{A} \otimes \boldsymbol{B})(\boldsymbol{C} \otimes \boldsymbol{D}) = (\boldsymbol{AC}) \otimes (\boldsymbol{BD}), \tag{S12.03}$$

for $\boldsymbol{A}$ and $\boldsymbol{B}$ as above, and $\boldsymbol{C}$ and $\boldsymbol{D}$, respectively, $q \times t$ and $s \times u$. We have that $\boldsymbol{A} \otimes \boldsymbol{B}$ is $pr \times qs$, $\boldsymbol{C} \otimes \boldsymbol{D}$ is $qs \times tu$, $\boldsymbol{AC}$ exists and is $p \times t$, and $\boldsymbol{BD}$ exists and is $r \times u$. Thus $(\boldsymbol{A} \otimes \boldsymbol{B})(\boldsymbol{C} \otimes \boldsymbol{D})$ exists and is $pr \times tu$, while $(\boldsymbol{AC}) \otimes (\boldsymbol{BD})$ is $pr \times tu$. Thus all the dimensions are correct.

Write out the left-hand side of equation (S12.03) explicitly as

$$\begin{bmatrix} a_{11}\boldsymbol{B} & \cdots & a_{1q}\boldsymbol{B} \\ \vdots & \ddots & \vdots \\ a_{p1}\boldsymbol{B} & \cdots & a_{pq}\boldsymbol{B} \end{bmatrix} \begin{bmatrix} c_{11}\boldsymbol{D} & \cdots & c_{1t}\boldsymbol{D} \\ \vdots & \ddots & \vdots \\ c_{q1}\boldsymbol{D} & \cdots & c_{qt}\boldsymbol{D} \end{bmatrix}.$$

The partitioning of both factors is compatible for multiplication, and we see directly that the product can be written as

$$\begin{bmatrix} \sum_{j=1}^q a_{1j}\, c_{j1}\boldsymbol{BD} & \cdots & \sum_{j=1}^q a_{1j}\, c_{jt}\boldsymbol{BD} \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^q a_{pj}\, c_{j1}\boldsymbol{BD} & \cdots & \sum_{j=1}^q a_{pj}\, c_{jt}\boldsymbol{BD} \end{bmatrix}. \tag{S12.04}$$

But $\sum_{j=1}^q a_{ij} c_{jk}$ is element (i,k) of $\boldsymbol{AC}$, and so it is clear that the right-hand side of equation (S12.04) is just $(\boldsymbol{AC}) \otimes (\boldsymbol{BD})$.

The third result to be shown is that

$$(\boldsymbol{A} \otimes \boldsymbol{B})^{-1} = \boldsymbol{A}^{-1} \otimes \boldsymbol{B}^{-1},$$

where $\boldsymbol{A}$ is $p \times p$ and $\boldsymbol{B}$ is $r \times r$, both nonsingular. Form the $pr \times pr$ product $(\boldsymbol{A} \otimes \boldsymbol{B})(\boldsymbol{A}^{-1} \otimes \boldsymbol{B}^{-1})$. By the previous result, this product is

$$\boldsymbol{A}\boldsymbol{A}^{-1} \otimes \boldsymbol{B}\boldsymbol{B}^{-1} = \mathbf{I}_p \otimes \mathbf{I}_r .$$

If the last expression is written out explicitly, we obtain the partitioned matrix

$$\begin{bmatrix} \mathbf{I}_r & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_r & \cdots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{O} & \mathbf{O} & \cdots & \mathbf{I}_r \end{bmatrix},$$

with p rows and columns of blocks. This matrix is manifestly just $\mathbf{I}_{pr}$, which shows that $\boldsymbol{A}^{-1} \otimes \boldsymbol{B}^{-1}$ is the inverse of $\boldsymbol{A} \otimes \boldsymbol{B}$.

Finally, we can answer the last part of the question. Since $a_{ij}b_{kl}$ is both element $((i-1)r+k, (j-1)s+l)$ of $\boldsymbol{A}\otimes\boldsymbol{B}$ and element $((k-1)p+i, (l-1)q+j)$ of $\boldsymbol{B}\otimes\boldsymbol{A}$, in order to go from $\boldsymbol{A}\otimes\boldsymbol{B}$ to $\boldsymbol{B}\otimes\boldsymbol{A}$, we must move row $(i-1)r+k$ of the former to row $(k-1)p+i$ for all i and k within the dimensions, and then move column $(j-1)s+l$ of the former to column $(l-1)q+j$ for all j and l within the dimensions. This prescription is unique, since, as i and k vary, $(i-1)r+k$ varies with no repetitions from 1 to pr, as does $(k-1)p+i$, with a similar result for the columns. This implies that we have defined a unique permutation of both rows and columns.