Solution to Exercise 12.13

⋆12.13 The **linear expenditure system** is a system of demand equations that can be written as

$$s_i = \frac{\gamma_i p_i}{E} + \alpha_i\Big(\frac{E - \sum_{j=1}^{m+1} p_j\gamma_j}{E}\Big). \tag{12.123}$$

Here, s_i, for $i = 1,\ldots,m$, is the share of total expenditure E spent on commodity i conditional on E and the prices p_i, for $i = 1,\ldots,m+1$. The equation indexed by $i = m+1$ is omitted as redundant, because the sum of the expenditure shares spent on all commodities is necessarily equal to 1. The model parameters are the α_i, $i = 1,\ldots,m$, the γ_i, $i = 1,\ldots,m+1$, and the $m \times m$ contemporaneous covariance matrix $\boldsymbol{\Sigma}$.

Express the system (12.123) as a linear SUR system by use of a suitable nonlinear reparametrization. The equations of the resulting system must be subject to a set of cross-equation restrictions. Express these restrictions in terms of the new parameters, and then set up a GNR in the manner of Section 12.3 that allows one to obtain restricted estimates of the α_i and γ_i.

Equation (12.123) is a special case of the linear system

$$s_i = \alpha_i + \sum_{j=1}^{m+1} \delta_{ij}(p_j/E). \tag{S12.16}$$

To obtain the linear expenditure system from (S12.16), we need to impose a large number of restrictions, since (S12.16) has $m + m(m+1) = m^2 + 2m$ parameters (not counting the parameters of the covariance matrix), while the linear expenditure system has only $2m + 1$. Thus we must impose $m^2 - 1$ restrictions on (S12.16).

Comparing (12.123) with (S12.16), we see that

$$\begin{aligned} &\delta_{ii} = (1-\alpha_i)\gamma_i, \quad i = 1,\ldots,m, \ \text{and} \\ &\delta_{ij} = -\alpha_i\gamma_j, \quad i = 1,\ldots,m, \ j = 1,\ldots,m+1, j \neq i. \end{aligned} \tag{S12.17}$$

These equations imply a set of nonlinear restrictions on the δ_{ij} and the α_i, as can be seen by eliminating the parameters γ_i that appear only in the restricted parametrization. We have

$$\gamma_i = \delta_{ii}/(1-\alpha_i) \text{ for } i = 1,\ldots,m, \text{ and } \gamma_{m+1} = -\delta_{1,m+1}/\alpha_1, \tag{S12.18}$$

where we have arbitrarily chosen to use the expression for $\delta_{1,m+1}$ to get an expression for γ_{m+1}. The first m equations of (S12.18) are equivalent to the m equations in the first line of (S12.17), while the last equation of (S12.18) is equivalent to one of the equations in the second line of (S12.17). Thus the $m+1$ equations of (S12.18) impose no restrictions. The second line of (S12.17) contains m^2 equations, of which only one has been used. The remaining m^2-1 equations are thus the m^2-1 restrictions needed to convert the unrestricted system (S12.16) into the restricted system (12.123).

The restrictions can be written explicitly in terms of the α_i and the δ_{ij} as follows. For $i = 1,\ldots,m$ and $j = 1,\ldots,m$, $j \neq i$, we have the m^2-m restrictions $\delta_{ij} = -\alpha_i\delta_{jj}/(1-\alpha_j)$, and, for $i = 2,\ldots,m$, we have the $m-1$ restrictions $\alpha_1\delta_{i,m+1} = \alpha_i\delta_{1,m+1}$.

For the GNR, we need the matrix of derivatives of the right-hand side of equation (12.123) with respect to the parameters. We find that

$$\begin{aligned}
\frac{\partial s_i}{\partial \alpha_i} &= 1 - \sum_{j=1}^{m+1} \gamma_j (p_j/E), \\
\frac{\partial s_i}{\partial \gamma_i} &= (1-\alpha_i)(p_i/E), \text{ and} \\
\frac{\partial s_i}{\partial \gamma_j} &= -\alpha_j (p_j/E), \;\; j \neq i.
\end{aligned}$$

These derivatives define the matrix $\boldsymbol{X}_i(\boldsymbol{\beta})$. We can then stack the m $\boldsymbol{X}_i(\boldsymbol{\beta})$ matrices to form $\boldsymbol{X}_\bullet(\boldsymbol{\beta})$, which has $2m+1$ columns and nm rows. Similarly, we stack the vectors of observations on the shares to form $\boldsymbol{y}_\bullet$ and the right-hand sides of (12.123) to form $\boldsymbol{x}_\bullet(\boldsymbol{\beta})$. The elements of the $(2m+1)$–vector $\boldsymbol{\beta}$ are the α_i and the γ_i.

We are now in a position to run the GNR (12.53). We first estimate the unrestricted model (S12.16) by OLS. This provides consistent estimates $\acute{\alpha}_i$ and $\acute{\delta}_{ij}$. To obtain consistent estimates of the γ_j, for $i = 1,\ldots,m$, the obvious approach is to use equations (S12.18). The OLS residuals from (S12.16) also allow us to estimate the contemporaneous covariance matrix $\boldsymbol{\Sigma}$, and thence to obtain a matrix $\acute{\boldsymbol{\Psi}}$ such that $\acute{\boldsymbol{\Psi}}\acute{\boldsymbol{\Psi}}^\top = \acute{\boldsymbol{\Sigma}}^{-1}$. If we run the GNR (12.53), with everything evaluated at these estimates, we can obtain one-step efficient estimates $\grave{\boldsymbol{\beta}} = \acute{\boldsymbol{\beta}} + \acute{\boldsymbol{b}}$. At this point, we have three choices:

1. We could stop with the one-step estimates $\grave{\boldsymbol{\beta}}$.
2. We could run the GNR again, still using $\acute{\boldsymbol{\Psi}}$, but evaluating $\boldsymbol{x}_\bullet(\boldsymbol{\beta})$ and $\boldsymbol{X}_\bullet(\boldsymbol{\beta})$ at $\grave{\boldsymbol{\beta}}$, and continue iterating until convergence. If this procedure converges, it yields feasible GLS estimates.
3. We could run the GNR again, computing a new matrix $\grave{\boldsymbol{\Psi}}$ in the obvious way and evaluating $\boldsymbol{x}_\bullet(\boldsymbol{\beta})$ and $\boldsymbol{X}_\bullet(\boldsymbol{\beta})$ at $\grave{\boldsymbol{\beta}}$, and continue iterating until convergence. If this procedure converges, it yields ML estimates.

Of course, the two iterative procedures need not converge. If not, modified versions in which the change in $\boldsymbol{\beta}$ from one iteration to the next is reduced by multiplying $\acute{\boldsymbol{b}}$ by a positive scalar less than unity might well do so.