Solution to Exercise 12.10

⋆12.10 Consider a multivariate linear regression model of the form (12.28) in which the regressors may include lags of the dependent variables and the error terms are normally distributed. By ordering the data appropriately, show that the determinant of the Jacobian of the transformation (12.31) is equal to unity. Then explain why this implies that the loglikelihood function, conditional on pre-sample observations, can be written as (12.33).

In equation (12.28) for the entire system, the data are ordered by equation, with all the observations for equation 1 followed by all the observations for equation 2, and so on. With this ordering, the Jacobian of the transformation is not in general triangular if there are lagged dependent variables. So as to obtain a triangular structure for the Jacobian, we want to reorder the data by observation and then by equation, with observation 1 for each equation in turn followed by observation 2 for each equation, and so on. Specifically, observation sj follows observation ti in this new ordering if and only if either $s > t$ or $s = t$ and $j > i$.

The row of equation (12.28) that corresponds to observation t and equation i can be rewritten as

$$u_{ti} = y_{ti} - \boldsymbol{Z}_{ti}\gamma_i - \sum_{j=1}^{p}\sum_{l=1}^{g}\delta_{jl}\,y_{t-j,l}, \tag{S12.12}$$

where no dependent variable appears lagged by more than p periods. For what follows, it does not matter if zero restrictions are imposed on the δ_{jl}, and so we do not take explicit account of this possibility. The k_i–vector $\boldsymbol{Z}_{ti}$ contains the exogenous explanatory variables for observation t in equation i. The equations (S12.12) relate the u_{ti} to the y_{sj}, $t, s = 1, \ldots, n$, $i, j = 1, \ldots, g$, conditional on all pre-sample information.

It is clear from (S12.12) that $\partial u_{ti}/\partial y_{ti} = 1$. Further, $\partial u_{ti}/\partial y_{tj} = 0$ for $j \neq i$, and $\partial u_{ti}/\partial y_{sj} = 0$ for $s > t$, $i, j = 1, \ldots, g$. These facts imply that, whenever observation sj follows observation ti in the sense defined above, $\partial u_{ti}/\partial y_{sj} = 0$. Thus the $gn \times gn$ Jacobian matrix for the transformation that gives the u_{ti} in terms of the y_{sj}, with the new ordering, is lower triangular with all its diagonal elements equal to 1. As in the previous exercise, therefore, the determinant of the Jacobian is unity.

Because the Jacobian factor is 1, the loglikelihood function for a system of linear regressions with lagged dependent variables looks just like the one for a system of linear regressions without lagged dependent variables. It is given by expression (12.33).