Solution to Exercise 11.9

⋆11.9 Plot $\Upsilon_t(\boldsymbol{\beta})$, which is defined in equation (11.16), as a function of $\boldsymbol{X}_t\boldsymbol{\beta}$ for both the logit and probit models. For the logit model only, prove that $\Upsilon_t(\boldsymbol{\beta})$ achieves its maximum value when $\boldsymbol{X}_t\boldsymbol{\beta} = 0$ and declines monotonically as $|\boldsymbol{X}_t\boldsymbol{\beta}|$ increases.

For the logit model, using the fact that $\Lambda(-x) = 1 - \Lambda(x)$, we see that

$$\Upsilon_t(\boldsymbol{\beta}) = \frac{\lambda^2(\boldsymbol{X}_t\boldsymbol{\beta})}{\Lambda(\boldsymbol{X}_t\boldsymbol{\beta})\Lambda(-\boldsymbol{X}_t\boldsymbol{\beta})}.$$

Using the relationship between $\lambda(x)$ and $\Lambda(x)$ given in equations (11.08), this reduces to

$$\Upsilon_t(\boldsymbol{\beta}) = \frac{\Lambda^2(\boldsymbol{X}_t\boldsymbol{\beta})\Lambda^2(-\boldsymbol{X}_t\boldsymbol{\beta})}{\Lambda(\boldsymbol{X}_t\boldsymbol{\beta})\Lambda(-\boldsymbol{X}_t\boldsymbol{\beta})} = \Lambda(\boldsymbol{X}_t\boldsymbol{\beta})\Lambda(-\boldsymbol{X}_t\boldsymbol{\beta}) = \lambda(\boldsymbol{X}_t\boldsymbol{\beta}).$$

Since

$$\lambda(x) = \frac{e^x}{(1+e^x)^2},$$

a little algebra shows that the first derivative of $\lambda(x)$ is

$$\lambda'(x) = \frac{e^x - e^{2x}}{(1+e^x)^3}. \tag{S11.09}$$

Because $e^{2x} = e^x$ when $x = 0$, we see that the derivative of $\lambda(x)$ is 0 when $x = 0$. Because $e^{2x} > e^x$ for $x > 0$ and $e^{2x} < e^x$ for $x < 0$, this derivative must be positive whenever $x < 0$ and negative whenever $x > 0$. Thus, because the denominator of (S11.09) is always positive, it follows that $\lambda(x)$ must achieve a maximum at $x = 0$ and decline monotonically as $|x|$ increases. What is true for $\lambda(x)$ is also true for $\Upsilon_t(\boldsymbol{\beta})$ regarded as a function of $\boldsymbol{X}_t\boldsymbol{\beta}$.

Unfortunately, no similarly simple proof appears to be available for the probit model. But the properties of $\Upsilon_t(\boldsymbol{\beta})$ that hold for the logit model also hold for the probit model, as Figure S11.1 shows. In both cases, the maximum is achieved at $\boldsymbol{X}_t\boldsymbol{\beta} = 0$, and the weights decline monotonically towards zero as $|\boldsymbol{X}_t\boldsymbol{\beta}|$ increases.

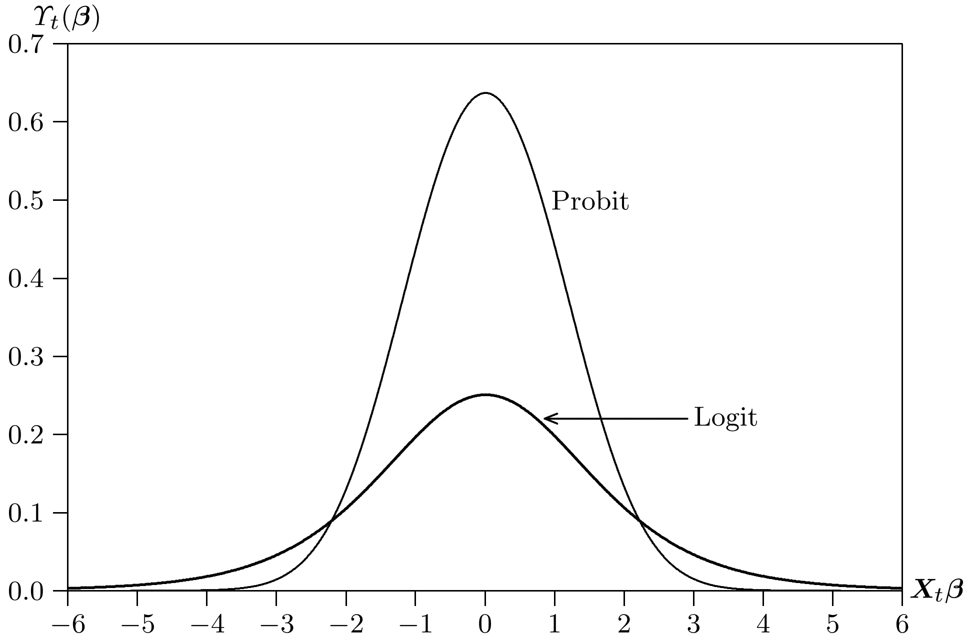

Figure S11.1 Weight functions for logit and probit models