Solution to Exercise 11.19

⋆**11.19** Let the one-step estimator $\grave{\boldsymbol{\theta}}$ be defined as usual for the discrete choice artificial regression (11.42) evaluated at a root-n consistent estimator $\acute{\boldsymbol{\theta}}$ as $\grave{\boldsymbol{\theta}} = \acute{\boldsymbol{\theta}} + \acute{\boldsymbol{b}}$, where $\acute{\boldsymbol{b}}$ is the vector of OLS parameter estimates from (11.42). Show that $\grave{\boldsymbol{\theta}}$ is asymptotically equivalent to the MLE $\hat{\boldsymbol{\theta}}$.

In order to prove this result, we need to show that the OLS estimates $\acute{\boldsymbol{b}}$ from the DCAR (11.42) are such that

$$n^{1/2}(\hat{\boldsymbol{\theta}} - \acute{\boldsymbol{\theta}}) \overset{a}{=} n^{1/2}\acute{\boldsymbol{b}}. \tag{S11.21}$$

In other words, we must show that $\acute{\boldsymbol{b}}$ is asymptotically equivalent to the difference between the MLE and the initial consistent estimator $\acute{\boldsymbol{\theta}}$. Previous results imply that

$$n^{1/2}\acute{\boldsymbol{b}} = \big(n^{-1}\boldsymbol{I}(\acute{\boldsymbol{\theta}})\big)^{-1} n^{-1/2}\boldsymbol{g}(\acute{\boldsymbol{\theta}}), \tag{S11.22}$$

where $\boldsymbol{g}(\acute{\boldsymbol{\theta}})$ denotes the gradient of the loglikelihood function (11.41) evaluated at $\acute{\boldsymbol{\theta}}$. In the previous exercise, we showed that the matrix of scalar products of the DCAR regressors with themselves is equal to the information matrix. In the text, we showed that the scalar product of any one of the regressors with the regressand is equal to (11.43) and that this is the derivative of (11.41) with respect to the corresponding parameter. Therefore, the vector of scalar products of the regressors with the regressand is equal to the gradient. Thus we have the result that $\boldsymbol{b}(\boldsymbol{\theta}) = \boldsymbol{I}^{-1}(\boldsymbol{\theta})\boldsymbol{g}(\boldsymbol{\theta})$. Evaluating this equation at $\acute{\boldsymbol{\theta}}$ and inserting the factors of powers of n that are required for asymptotic analysis yields equation (S11.22).

The first-order conditions that determine the MLE $\hat{\boldsymbol{\theta}}$ are simply $\boldsymbol{g}(\hat{\boldsymbol{\theta}}) = \mathbf{0}$. If we perform a first-order Taylor expansion of these equations around $\acute{\boldsymbol{\theta}}$, we obtain

$$\boldsymbol{g}(\acute{\boldsymbol{\theta}}) + \boldsymbol{H}(\acute{\boldsymbol{\theta}})(\hat{\boldsymbol{\theta}} - \acute{\boldsymbol{\theta}}) \overset{a}{=} \mathbf{0}.$$

Solving these for $\hat{\boldsymbol{\theta}} - \acute{\boldsymbol{\theta}}$ yields

$$\hat{\boldsymbol{\theta}} - \acute{\boldsymbol{\theta}} \overset{a}{=} -\boldsymbol{H}^{-1}(\acute{\boldsymbol{\theta}})\boldsymbol{g}(\acute{\boldsymbol{\theta}}).$$

When the appropriate factors of powers of n are inserted, this becomes

$$\begin{aligned} n^{1/2}(\hat{\boldsymbol{\theta}} - \acute{\boldsymbol{\theta}}) &\overset{a}{=} -\big(n^{-1}\boldsymbol{H}(\acute{\boldsymbol{\theta}})\big)^{-1} n^{-1/2}\boldsymbol{g}(\acute{\boldsymbol{\theta}}) \\ &\overset{a}{=} \big(n^{-1}\boldsymbol{I}(\acute{\boldsymbol{\theta}})\big)^{-1} n^{-1/2}\boldsymbol{g}(\acute{\boldsymbol{\theta}}). \end{aligned}$$

The second line here is the right-hand side of (S11.22). Therefore, we have shown that (S11.21) holds. This establishes the asymptotic equivalence of the one-step estimator $\grave{\boldsymbol{\theta}}$ and the ML estimator $\hat{\boldsymbol{\theta}}$.