Solution to Exercise 11.14

⋆11.14 Show, by use of l'Hôpital's rule or otherwise, that the two results in (11.29) hold for all functions $\tau(\cdot)$ which satisfy conditions (11.28).

The first result is that

$$\lim_{\delta\to 0}\left(\frac{\tau(\delta x)}{\delta}\right) = x.$$

Since both numerator and denominator equal 0 when $\delta = 0$, we can use l'Hôpital's rule. The derivative of the numerator is $\tau'(\delta x)x$, and that of the denominator is 1. Evaluating the numerator at $\delta = 0$, we find that, since $\tau'(0) = 1$, it is just equal to x. Since $x/1 = x$, we have proved the first result.

The second result is that

$$\lim_{\delta\to 0}\left(\frac{\partial\big(\tau(\delta x)/\delta\big)}{\partial\delta}\right) = \tfrac{1}{2}x^2\tau''(0).$$

The derivative of $\tau(\delta x)/\delta$ with respect to δ is

$$\frac{\tau'(\delta x)x\delta - \tau(\delta x)}{\delta^2}.$$

Once again, both numerator and denominator equal 0 when $\delta = 0$. This time, we need to take derivatives twice in order to apply l'Hôpital's rule. The derivative of the numerator is

$$\tau''(\delta x)x^2\delta + \tau'(\delta x)x - \tau'(\delta x)x = \tau''(\delta x)x^2\delta,$$

which is once again equal to 0 when $\delta = 0$. So is 2δ, the derivative of the denominator. Differentiating each of them again, as l'Hôpital's rule tells us to do, we find that the second derivative of the numerator is

$$\tau'''(\delta x)x^3\delta + \tau''(\delta x)x^2,$$

and the second derivative of the denominator is just 2. When $\delta = 0$, the first term in the derivative of the numerator vanishes, and we are left with $\tau''(0)x^2$, which by the third condition in (11.28), is nonzero. Dividing this by 2, we find that the limit is just $\frac{1}{2}\tau''(0)x^2$, which is what we were required to show.