and GARCH models discussed here, among them the **exponential GARCH** model of Nelson (1991) and the **absolute GARCH** model of Hentschel (1995). These models are intended to explain empirical features of financial time series that the standard GARCH model cannot capture. More detailed treatments may be found in Bollerslev, Chou, and Kroner (1992), Bollerslev, Engle, and Nelson (1994), Hamilton (1994, Chapter 21), and Pagan (1996).

13.7 Vector Autoregressions

The dynamic models discussed in Section 13.4 were single-equation models. But we often want to model the dynamic relationships among several time-series variables. A simple way to do so without making many assumptions is to use what is called a **vector autoregression**, or **VAR**, model, which is the multivariate analog of an autoregressive model for a single time series.

Let the $1\times g$ vector $\boldsymbol{Y}_t$ denote the t^{th} observation on a set of g variables. Then a vector autoregressive model of order p, sometimes referred to as a **VAR($\boldsymbol{p}$) model**, can be written as

$$\boldsymbol{Y}_t = \boldsymbol{\alpha} + \sum_{j=1}^{p} \boldsymbol{Y}_{t-j}\,\boldsymbol{\Phi}_j + \boldsymbol{U}_t, \quad \boldsymbol{U}_t \sim \mathrm{IID}(\boldsymbol{0}, \boldsymbol{\Sigma}), \tag{13.87}$$

where $\boldsymbol{U}_t$ is a $1\times g$ vector of error terms, $\boldsymbol{\alpha}$ is a $1\times g$ vector of constant terms, and the $\boldsymbol{\Phi}_j$, for $j = 1,\ldots,p$, are $g\times g$ matrices of coefficients, all of which are to be estimated. If y_{ti} denotes the i^{th} element of $\boldsymbol{Y}_t$ and $\phi_{j,ki}$ denotes the ki^{th} element of $\boldsymbol{\Phi}_j$, then the i^{th} column of (13.87) can be written as

$$y_{ti} = \alpha_i + \sum_{j=1}^{p}\sum_{k=1}^{g} y_{t-j,k}\,\phi_{j,ki} + u_{ti}.$$

This is just a linear regression, in which y_{ti} depends on a constant term and lags 1 through p of all of the g variables in the system. Thus we see that the VAR (13.87) has the form of a multivariate linear regression model, or SUR model, like the ones we discussed in Section 12.2.

To see this clearly, let us make the definitions

$$\boldsymbol{X}_t \equiv [1 \quad \boldsymbol{Y}_{t-1} \quad \cdots \quad \boldsymbol{Y}_{t-p}] \quad \text{and} \quad \boldsymbol{\Pi} \equiv \begin{bmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\Phi}_1 \\ \vdots \\ \boldsymbol{\Phi}_p \end{bmatrix}.$$

The row vector $\boldsymbol{X}_t$ has $k \equiv gp+1$ elements, and the matrix $\boldsymbol{\Pi}$ is $k\times g$. With these definitions, the VAR (13.87) becomes

$$\boldsymbol{Y}_t = \boldsymbol{X}_t\boldsymbol{\Pi} + \boldsymbol{U}_t, \quad \boldsymbol{U}_t \sim \mathrm{IID}(\boldsymbol{0}, \boldsymbol{\Sigma}), \tag{13.88}$$