where we suppress the dependence on $\boldsymbol{y}$ for notational simplicity. The notation $\bar{\boldsymbol{\theta}}$ is our usual shorthand notation for Taylor expansions of vector expressions; see (6.20) and the subsequent discussion. We may therefore write

$$\big\|\bar{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\big\| \leq \big\|\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\big\|.$$

The fact that the ML estimator $\hat{\boldsymbol{\theta}}$ is consistent then implies that $\bar{\boldsymbol{\theta}}$ is also consistent.

If we solve (10.35) and insert the factors of powers of $n$ that are needed for asymptotic analysis, we obtain the result that

$$n^{1/2}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) = -\big(n^{-1}\boldsymbol{H}(\bar{\boldsymbol{\theta}})\big)^{-1}\big(n^{-1/2}\boldsymbol{g}(\boldsymbol{\theta}_0)\big). \tag{10.36}$$

Because $\bar{\boldsymbol{\theta}}$ is consistent, the matrix $n^{-1}\boldsymbol{H}(\bar{\boldsymbol{\theta}})$ which appears in (10.36) must tend to the same nonstochastic limiting matrix as $n^{-1}\boldsymbol{H}(\boldsymbol{\theta}_0)$, namely, $\mathcal{H}(\boldsymbol{\theta}_0)$. Therefore, equation (10.36) implies that

$$n^{1/2}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \overset{a}{=} -\mathcal{H}^{-1}(\boldsymbol{\theta}_0)n^{-1/2}\boldsymbol{g}(\boldsymbol{\theta}_0). \tag{10.37}$$

If the information matrix equality, equation (10.34), holds, then this result can equivalently be written as

$$n^{1/2}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \overset{a}{=} \mathcal{I}^{-1}(\boldsymbol{\theta}_0)n^{-1/2}\boldsymbol{g}(\boldsymbol{\theta}_0). \tag{10.38}$$

Since the information matrix equality holds only if the model is correctly specified, (10.38) is not in general valid for misspecified models.

The asymptotic normality of the Type 2 MLE follows immediately from the asymptotic equalities (10.37) or (10.38) if it can be shown that the vector $n^{-1/2}\boldsymbol{g}(\boldsymbol{\theta}_0)$ is asymptotically distributed as multivariate normal. As can be seen from (10.27), each element $n^{-1/2}g_i(\boldsymbol{\theta}_0)$ of this vector is $n^{-1/2}$ times a sum of $n$ random variables, each of which has mean 0, by (10.29). These random variables are mutually uncorrelated, by the result (10.30). Under standard regularity conditions, with which we will not concern ourselves, a multivariate central limit theorem can therefore be applied to this vector. For finite $n$, the covariance matrix of the score vector is, by definition, the information matrix $\boldsymbol{I}(\boldsymbol{\theta}_0)$. Thus the covariance matrix of the vector $n^{-1/2}\boldsymbol{g}(\boldsymbol{\theta}_0)$ is $n^{-1}\boldsymbol{I}(\boldsymbol{\theta}_0)$, of which, by (10.32), the limit as $n \to \infty$ is the asymptotic information matrix $\mathcal{I}(\boldsymbol{\theta}_0)$. It follows that

$$n^{-1/2}\boldsymbol{g}(\boldsymbol{\theta}_0) \overset{a}{\sim} \mathrm{N}\big(\mathbf{0}, \mathcal{I}(\boldsymbol{\theta}_0)\big). \tag{10.39}$$

This result, when combined with (10.37) or (10.38), implies that the Type 2 MLE is asymptotically normally distributed.