The denominator of (4.51) is, thankfully, easier to analyze. The square of the second factor is

$$\begin{aligned} n^{-1}\boldsymbol{x}_2^{\top}\boldsymbol{M}_1\boldsymbol{x}_2 &= n^{-1}\boldsymbol{x}_2^{\top}\boldsymbol{x}_2 - n^{-1}\boldsymbol{x}_2^{\top}\boldsymbol{P}_1\boldsymbol{x}_2 \\ &= n^{-1}\boldsymbol{x}_2^{\top}\boldsymbol{x}_2 - n^{-1}\boldsymbol{x}_2^{\top}\boldsymbol{X}_1\big(n^{-1}\boldsymbol{X}_1^{\top}\boldsymbol{X}_1\big)^{-1}n^{-1}\boldsymbol{X}_1^{\top}\boldsymbol{x}_2. \end{aligned}$$

In the limit, all the pieces of this expression become submatrices of $\boldsymbol{S}_{\boldsymbol{X}^{\top}\boldsymbol{X}}$, and so we find that

$$n^{-1}\boldsymbol{x}_2^{\top}\boldsymbol{M}_1\boldsymbol{x}_2 \to \boldsymbol{S}_{22} - \boldsymbol{S}_{21}\boldsymbol{S}_{11}^{-1}\boldsymbol{S}_{12}.$$

When it is multiplied by σ_0^2, this is just (4.57), the variance of the numerator of expression (4.51). Thus, asymptotically, we have shown that t_{β_2} is the ratio of a normal random variable with mean zero to the standard deviation of that random variable. Consequently, we have established that, under the null hypothesis, with regressors that are not necessarily exogenous but merely predetermined, $t_{\beta_2} \overset{a}{\sim} \mathrm{N}(0,1)$. This result is what we previously obtained as (4.52) when we assumed that the regressors were exogenous.

Asymptotic F Tests

A similar analysis can be performed for the F statistic (4.33) for the null hypothesis that $\boldsymbol{\beta}_2 = \boldsymbol{0}$ in the model (4.28). Under the null, $F_{\boldsymbol{\beta}_2}$ is equal to expression (4.34), which can be rewritten as

$$\frac{n^{-1/2}\boldsymbol{\varepsilon}^{\top}\boldsymbol{M}_1\boldsymbol{X}_2(n^{-1}\boldsymbol{X}_2^{\top}\boldsymbol{M}_1\boldsymbol{X}_2)^{-1}n^{-1/2}\boldsymbol{X}_2^{\top}\boldsymbol{M}_1\boldsymbol{\varepsilon}/r}{\boldsymbol{\varepsilon}^{\top}\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{\varepsilon}/(n-k)}, \tag{4.58}$$

where $\boldsymbol{\varepsilon} \equiv \boldsymbol{u}/\sigma_0$ and $r = k_2$, the dimension of $\boldsymbol{\beta}_2$. It is not hard to use the results we obtained for the t statistic to show that, as $n \to \infty$,

$$rF_{\boldsymbol{\beta}_2} \overset{a}{\sim} \chi^2(r) \tag{4.59}$$

under the null hypothesis; see Exercise 4.13. Since $1/r$ times a random variable that follows the $\chi^2(r)$ distribution is distributed as $F(r,\infty)$, we can also conclude that $F_{\boldsymbol{\beta}_2} \overset{a}{\sim} F(r, n-k)$.

The results (4.52) and (4.59) justify the use of t and F tests outside the confines of the classical normal linear model. We can compute P values using either the standard normal or t distributions in the case of t statistics, and either the χ^2 or F distributions in the case of F statistics. Of course, if we use the χ^2 distribution, we have to multiply the F statistic by r.

Whatever distribution we use, these P values are approximate, and tests based on them are not exact in finite samples. In addition, our theoretical results do not tell us just how accurate they are. If we decide to use a nominal level of α for a test, we reject if the approximate P value is less than α. In many cases, but certainly not all, such tests are probably quite accurate,