is **asymptotically distributed** as $N(0, 1)$. This means that, as $n \to \infty$, the random variable z tends to a random variable which follows the $N(0, 1)$ distribution. It may seem curious that we divide by $\sqrt{n}$ instead of by n in (4.45), but this is an essential feature of every CLT. To see why, we calculate the variance of z. Since the terms in the sum in (4.45) are independent, the variance of z is just the sum of the variances of the n terms:

$$\mathrm{Var}(z) = n\,\mathrm{Var}\Big(\frac{1}{\sqrt{n}}\frac{x_t - \mu}{\sigma}\Big) = \frac{n}{n} = 1.$$

If we had divided by n, we would, by a law of large numbers, have obtained a random variable with a plim of 0 instead of a random variable with a limiting standard normal distribution. Thus, whenever we want to use a CLT, we must ensure that a factor of $n^{-1/2} = 1/\sqrt{n}$ is present.

Just as there are many different LLNs, so too are there many different CLTs, almost all of which impose weaker conditions on the x_t than those imposed by the Lindeberg-Lévy CLT. The assumption that the x_t are identically distributed is easily relaxed, as is the assumption that they are independent. However, if there is either too much dependence or too much heterogeneity, a CLT may not apply. Several CLTs are discussed in Section 4.7 of Davidson and MacKinnon (1993), and Davidson (1994) provides a more advanced treatment. In all cases of interest to us, the CLT says that, for a sequence of uncorrelated random variables x_t, $t = 1, \ldots, \infty$, with $\mathrm{E}(x_t) = 0$,

$$\operatorname*{plim}_{n\to\infty} n^{-1/2}\sum_{t=1}^{n} x_t = x_0 \sim N\Big(0, \lim_{n\to\infty}\tfrac{1}{n}\sum_{t=1}^{n}\mathrm{Var}(x_t)\Big).$$

We sometimes need vector, or **multivariate**, versions of CLTs. Suppose that we have a sequence of uncorrelated random m–vectors $\boldsymbol{x}_t$, for some fixed m, with $\mathrm{E}(\boldsymbol{x}_t) = \mathbf{0}$. Then the appropriate multivariate CLT tells us that

$$\operatorname*{plim}_{n\to\infty} n^{-1/2}\sum_{t=1}^{n} \boldsymbol{x}_t = \boldsymbol{x}_0 \sim N\Big(\mathbf{0}, \lim_{n\to\infty}\tfrac{1}{n}\sum_{t=1}^{n}\mathrm{Var}(\boldsymbol{x}_t)\Big), \tag{4.46}$$

where $\boldsymbol{x}_0$ is multivariate normal, and each $\mathrm{Var}(\boldsymbol{x}_t)$ is an $m \times m$ matrix.

Figure 4.7 illustrates the fact that CLTs often provide good approximations even when n is not very large. Both panels of the figure show the densities of various random variables z defined as in (4.45). In the top panel, the x_t are uniformly distributed, and we see that z is remarkably close to being distributed as standard normal even when n is as small as 8. This panel does not show results for larger values of n because they would have made it too hard to read. In the bottom panel, the x_t follow the $\chi^2(1)$ distribution, which exhibits extreme right skewness. The mode[6] of the distribution is 0, there

[6] A **mode** of a distribution is a point at which the density achieves a local maximum. If there is just one such point, a density is said to be **unimodal**.