To save space, we can also write this as $\boldsymbol{X} = \begin{bmatrix}\boldsymbol{X}_1 \vdots \boldsymbol{X}_2 \vdots \ldots \vdots \boldsymbol{X}_n\end{bmatrix}$. There is no restriction on how a matrix can be partitioned, so long as all the **submatrices** or **blocks** fit together correctly. Thus we might have

$$\boldsymbol{X} = \begin{array}{c} \\ \\ \end{array}\begin{array}{cc} k_1 & k_2 \\ \end{array}$$

$$\boldsymbol{X} = \begin{array}{cc} & \begin{array}{cc} \;k_1 & \;\;k_2 \end{array} \\ & \begin{bmatrix} \boldsymbol{X}_{11} & \boldsymbol{X}_{12} \\ \boldsymbol{X}_{21} & \boldsymbol{X}_{22} \end{bmatrix} \begin{array}{c} n_1 \\ n_2 \end{array} \end{array}$$

or, equivalently, $\boldsymbol{X} = [\boldsymbol{X}_{11} \vdots \boldsymbol{X}_{21} \quad \boldsymbol{X}_{12} \vdots \boldsymbol{X}_{22}]$. Here the submatrix $\boldsymbol{X}_{11}$ has dimensions $n_1 \times k_1$, $\boldsymbol{X}_{12}$ has dimensions $n_1 \times k_2$, $\boldsymbol{X}_{21}$ has dimensions $n_2 \times k_1$, and $\boldsymbol{X}_{22}$ has dimensions $n_2 \times k_2$, with $n_1 + n_2 = n$ and $k_1 + k_2 = k$. Thus $\boldsymbol{X}_{11}$ and $\boldsymbol{X}_{12}$ have the same number of rows, and also $\boldsymbol{X}_{21}$ and $\boldsymbol{X}_{22}$, as required for the submatrices to fit together horizontally. Similarly, $\boldsymbol{X}_{11}$ and $\boldsymbol{X}_{21}$ have the same number of columns, and also $\boldsymbol{X}_{12}$ and $\boldsymbol{X}_{22}$, as required for the submatrices to fit together vertically as well.

If two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ of the same dimensions are partitioned in exactly the same way, they can be added or subtracted block by block. A simple example is

$$\boldsymbol{A} + \boldsymbol{B} = [\,\boldsymbol{A}_1 \quad \boldsymbol{A}_2\,] + [\,\boldsymbol{B}_1 \quad \boldsymbol{B}_2\,] = [\,\boldsymbol{A}_1 + \boldsymbol{B}_1 \quad \boldsymbol{A}_2 + \boldsymbol{B}_2\,],$$

where $\boldsymbol{A}_1$ and $\boldsymbol{B}_1$ have the same dimensions, as do $\boldsymbol{A}_2$ and $\boldsymbol{B}_2$.

More interestingly, as we now explain, matrix multiplication can sometimes be performed block by block on partitioned matrices. If the product $\boldsymbol{AB}$ exists, then $\boldsymbol{A}$ has as many columns as $\boldsymbol{B}$ has rows. Now suppose that the columns of $\boldsymbol{A}$ are partitioned in the same way as the rows of $\boldsymbol{B}$. Then

$$\boldsymbol{AB} = [\,\boldsymbol{A}_1 \quad \boldsymbol{A}_2 \quad \cdots \quad \boldsymbol{A}_p\,] \begin{bmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \\ \vdots \\ \boldsymbol{B}_p \end{bmatrix}.$$

Here each $\boldsymbol{A}_i$, $i = 1, \ldots, p$, has as many columns as the corresponding $\boldsymbol{B}_i$ has rows. The product can be computed following the usual rules for matrix multiplication just as though the blocks were scalars, yielding the result

$$\boldsymbol{AB} = \sum_{i=1}^{p} \boldsymbol{A}_i \boldsymbol{B}_i. \tag{1.35}$$

To see this, it is enough to compute the typical element of each side of equation (1.35) directly and observe that they are the same. Matrix multiplication can also be performed block by block on matrices that are partitioned both horizontally and vertically, provided all the submatrices are conformable; see Exercise 1.19.