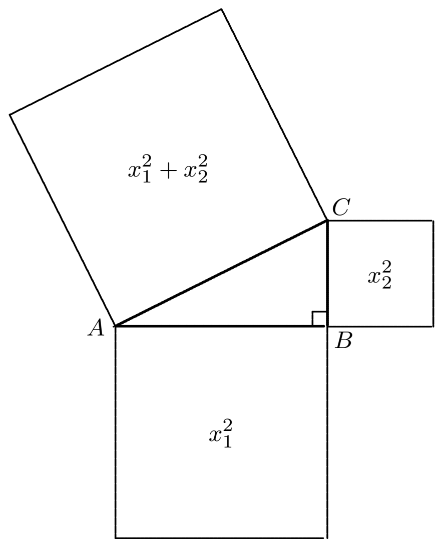

Figure 2.1 Pythagoras' Theorem

equal to the sum of the squares on the other two sides. This longest side is called the **hypotenuse**. Pythagoras' Theorem is illustrated in Figure 2.1. The figure shows a right-angled triangle, ABC, with hypotenuse AC and two other sides, AB and BC, of lengths x_1 and x_2, respectively. The squares on each of the three sides of the triangle are drawn, and the area of the square on the hypotenuse is shown as $x_1^2 + x_2^2$, in accordance with the theorem.

A beautiful proof of Pythagoras' Theorem, not often found in geometry texts, is shown in Figure 2.2. Two squares of equal area are drawn. Each square contains four copies of the same right-angled triangle. The square on the left also contains the squares on the two shorter sides of the triangle, while the

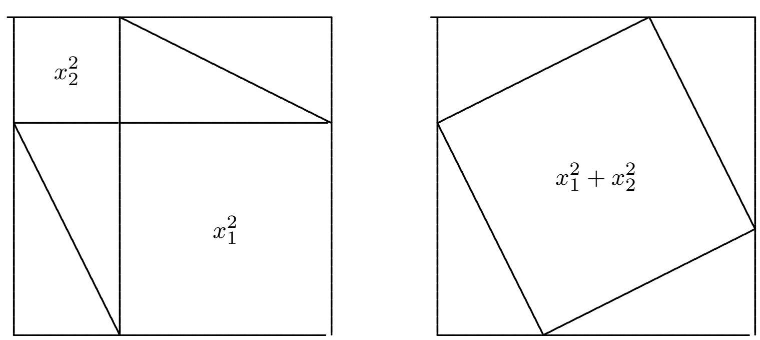

Figure 2.2 Proof of Pythagoras' Theorem

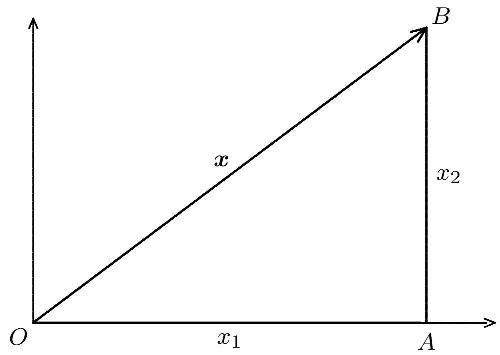

Figure 2.3 A vector $\boldsymbol{x}$ in E^2

square on the right contains the square on the hypotenuse. The theorem follows at once.

Any vector $\boldsymbol{x} \in E^2$ has two components, usually denoted as x_1 and x_2. These two components can be interpreted as the **Cartesian coordinates** of the vector in the plane. The situation is illustrated in Figure 2.3. With O as the origin of the coordinates, a right-angled triangle is formed by the lines OA, AB, and OB. The length of the horizontal side of the triangle, OA, is the horizontal coordinate x_1. The length of the vertical side, AB, is the vertical coordinate x_2. Thus the point B has Cartesian coordinates (x_1, x_2). The vector $\boldsymbol{x}$ itself is usually represented as the hypotenuse of the triangle, OB, that is, the directed line (depicted as an arrow) joining the origin to the point B, with coordinates (x_1, x_2). Pythagoras' Theorem tells us that the length of the vector $\boldsymbol{x}$, which is the hypotenuse of the triangle, is $(x_1^2 + x_2^2)^{1/2}$. By (2.03), this is what $\|\boldsymbol{x}\|$ is equal to when $n = 2$.

Vector Geometry in Two Dimensions

Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be two vectors in E^2, with components (x_1, x_2) and (y_1, y_2), respectively. Then, by the rules of matrix addition, the components of $\boldsymbol{x} + \boldsymbol{y}$ are $(x_1 + y_1, x_2 + y_2)$. Figure 2.4 shows how the addition of $\boldsymbol{x}$ and $\boldsymbol{y}$ can be performed geometrically in two different ways. The vector $\boldsymbol{x}$ is drawn as the directed line segment, or arrow, from the origin O to the point A with coordinates (x_1, x_2). The vector $\boldsymbol{y}$ can be drawn similarly and represented by the arrow OB. However, we could also draw $\boldsymbol{y}$ starting, not at O, but at the point reached after drawing $\boldsymbol{x}$, namely A. The arrow AC has the same length and direction as OB, and we will see in general that arrows with the same length and direction can be taken to represent the same vector. It is clear by construction that the coordinates of C are $(x_1 + y_1, x_2 + y_2)$, that is, the coordinates of $\boldsymbol{x} + \boldsymbol{y}$. Thus the sum $\boldsymbol{x} + \boldsymbol{y}$ is represented geometrically by the arrow OC.

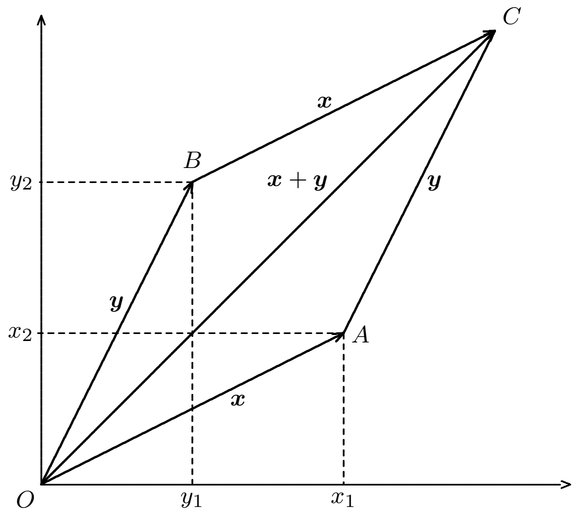

Figure 2.4 Addition of vectors

The classical way of adding vectors geometrically is to form a parallelogram using the line segments OA and OB that represent the two vectors as adjacent sides of the parallelogram. The sum of the two vectors is then the diagonal through O of the resulting parallelogram. It is easy to see that this classical method also gives the result that the sum of the two vectors is represented by the arrow OC, since the figure $OACB$ is just the parallelogram required by the construction, and OC is its diagonal through O. The parallelogram construction also shows clearly that vector addition is commutative, since $\boldsymbol{y}+\boldsymbol{x}$ is represented by OB, for $\boldsymbol{y}$, followed by BC, for $\boldsymbol{x}$. The end result is once more OC.

Multiplying a vector by a scalar is also very easy to represent geometrically. If a vector $\boldsymbol{x}$ with components (x_1, x_2) is multiplied by a scalar α, then $\alpha\boldsymbol{x}$ has components $(\alpha x_1, \alpha x_2)$. This is depicted in Figure 2.5, where $\alpha = 2$. The line segments OA and OB represent $\boldsymbol{x}$ and $\alpha\boldsymbol{x}$, respectively. It is clear that even if we move $\alpha\boldsymbol{x}$ so that it starts somewhere other than O, as with CD in the figure, the vectors $\boldsymbol{x}$ and $\alpha\boldsymbol{x}$ are always **parallel**. If α were negative, then $\alpha\boldsymbol{x}$ would simply point in the opposite direction. Thus, for $\alpha = -2$, $\alpha\boldsymbol{x}$ would be represented by DC, rather than CD.

Another property of multiplication by a scalar is clear from Figure 2.5. By direct calculation,

$$\|\alpha\boldsymbol{x}\| = \langle\alpha\boldsymbol{x}, \alpha\boldsymbol{x}\rangle^{1/2} = |\alpha|\,(\boldsymbol{x}^\top\boldsymbol{x})^{1/2} = |\alpha|\,\|\boldsymbol{x}\|. \qquad (2.04)$$

Since $\alpha = 2$, OB and CD in the figure are twice as long as OA.