Exact Confidence Regions for Regression Parameters

Suppose that we want to construct a confidence region for the elements of the vector $\boldsymbol{\beta}_2$ in the classical normal linear model (4.28), which we rewrite here for ease of exposition:

$$\boldsymbol{y} = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_2\boldsymbol{\beta}_2 + \boldsymbol{u}, \quad \boldsymbol{u} \sim \mathrm{N}(\mathbf{0}, \sigma^2\mathbf{I}), \tag{5.24}$$

where $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$ are a k_1–vector and a k_2–vector, respectively. The F statistic that can be used to test the hypothesis that $\boldsymbol{\beta}_2 = \mathbf{0}$ is given in (4.33). If we wish instead to test $\boldsymbol{\beta}_2 = \boldsymbol{\beta}_{20}$, then we can write (5.24) as

$$\boldsymbol{y} - \boldsymbol{X}_2\boldsymbol{\beta}_{20} = \boldsymbol{X}_1\boldsymbol{\beta}_1 + \boldsymbol{X}_2\boldsymbol{\gamma} + \boldsymbol{u}, \quad \boldsymbol{u} \sim \mathrm{N}(\mathbf{0}, \sigma^2\mathbf{I}), \tag{5.25}$$

where $\boldsymbol{\gamma} = \boldsymbol{\beta}_2 - \boldsymbol{\beta}_{20}$. We wish to test the hypothesis that $\boldsymbol{\gamma} = \mathbf{0}$. It is not hard to show that the F statistic for this hypothesis takes the form

$$\frac{(\hat{\boldsymbol{\beta}}_2 - \boldsymbol{\beta}_{20})^{\top}\boldsymbol{X}_2^{\top}\boldsymbol{M}_1\boldsymbol{X}_2(\hat{\boldsymbol{\beta}}_2 - \boldsymbol{\beta}_{20})/k_2}{\boldsymbol{y}^{\top}\boldsymbol{M}_{\boldsymbol{X}}\boldsymbol{y}/(n-k)}, \tag{5.26}$$

where $k = k_1 + k_2$; see Exercise 5.10. When multiplied by k_2, this F statistic is in the form of the Wald statistic (5.18). For the purposes of inference on $\boldsymbol{\beta}_2$, regression (5.24) is, by the FWL Theorem, equivalent to the regression

$$\boldsymbol{M}_1\boldsymbol{y} = \boldsymbol{M}_1\boldsymbol{X}_2\boldsymbol{\beta}_2 + \boldsymbol{M}_1\boldsymbol{u}.$$

Thus $\mathrm{Var}(\hat{\boldsymbol{\beta}}_2)$ is equal to $\sigma^2(\boldsymbol{X}_2^{\top}\boldsymbol{M}_1\boldsymbol{X}_2)^{-1}$. Since the denominator of (5.26) is just s^2, the OLS estimate of the error variance from running regression (5.24), k_2 times the F statistic (5.26) can be written in the form of (5.18), with

$$\widehat{\mathrm{Var}}(\hat{\boldsymbol{\beta}}_2) = s^2\big(\boldsymbol{X}_2^{\top}\boldsymbol{M}_1\boldsymbol{X}_2\big)^{-1}$$

providing a consistent estimator of the variance of $\hat{\boldsymbol{\beta}}_2$; compare (3.50).

Under the assumptions of the classical normal linear model, the F statistic (5.26) follows the $F(k_2, n-k)$ distribution when the null hypothesis is true. Therefore, we can use it to construct an **exact confidence region**. If c_α denotes the $1-\alpha$ quantile of the $F(k_2, n-k)$ distribution, then the $1-\alpha$ confidence region is the set of all $\boldsymbol{\beta}_{20}$ for which

$$(\hat{\boldsymbol{\beta}}_2 - \boldsymbol{\beta}_{20})^{\top}\boldsymbol{X}_2^{\top}\boldsymbol{M}_1\boldsymbol{X}_2(\hat{\boldsymbol{\beta}}_2 - \boldsymbol{\beta}_{20}) \le c_\alpha k_2 s^2. \tag{5.27}$$

Since the left-hand side of this inequality is quadratic in $\boldsymbol{\beta}_{20}$, the confidence region is, for $k_2 = 2$, the interior of an ellipse and, for $k_2 > 2$, the interior of a k_2–dimensional ellipsoid.