Because this matrix is triangular, its determinant is simply the product of the elements on the principal diagonal, which is 1. Therefore, there is no Jacobian term in the loglikelihood function (12.80) for such a system, and the ML estimates may be obtained by minimizing the determinant

$$\big|(\boldsymbol{Y} - \boldsymbol{W}\boldsymbol{B}\boldsymbol{\Gamma}^{-1})^{\top}(\boldsymbol{Y} - \boldsymbol{W}\boldsymbol{B}\boldsymbol{\Gamma}^{-1})\big| = \big|(\boldsymbol{Y}\boldsymbol{\Gamma} - \boldsymbol{W}\boldsymbol{B})^{\top}(\boldsymbol{Y}\boldsymbol{\Gamma} - \boldsymbol{W}\boldsymbol{B})\big|.$$

It can, with considerable effort, be shown that minimizing this determinant is equivalent to minimizing the ratio

$$\kappa \equiv \frac{(\boldsymbol{y} - \boldsymbol{Y}\boldsymbol{\beta}_2)^{\top}\boldsymbol{M}_{\boldsymbol{Z}}(\boldsymbol{y} - \boldsymbol{Y}\boldsymbol{\beta}_2)}{(\boldsymbol{y} - \boldsymbol{Y}\boldsymbol{\beta}_2)^{\top}\boldsymbol{M}_{\boldsymbol{W}}(\boldsymbol{y} - \boldsymbol{Y}\boldsymbol{\beta}_2)} = \frac{\boldsymbol{\gamma}^{\top}\boldsymbol{Y}_*^{\top}\boldsymbol{M}_{\boldsymbol{Z}}\boldsymbol{Y}_*\boldsymbol{\gamma}}{\boldsymbol{\gamma}^{\top}\boldsymbol{Y}_*^{\top}\boldsymbol{M}_{\boldsymbol{W}}\boldsymbol{Y}_*\boldsymbol{\gamma}} \tag{12.119}$$

with respect to $\boldsymbol{\beta}_2$, where $\boldsymbol{Y}_* \equiv [\boldsymbol{y} \;\; \boldsymbol{Y}]$ and $\boldsymbol{\gamma} = [1 \;\vdots\; -\boldsymbol{\beta}_2]$; see Davidson and MacKinnon (1993, Chapter 18).

It is possible to minimize κ without doing any sort of nonlinear optimization. The first-order conditions obtained by differentiating the middle expression in (12.119) with respect to $\boldsymbol{\beta}_2$ can be rearranged as

$$\boldsymbol{Y}^{\top}(\boldsymbol{M}_{\boldsymbol{Z}} - \hat{\kappa}\boldsymbol{M}_{\boldsymbol{W}})\boldsymbol{Y}_*\boldsymbol{\gamma} = \boldsymbol{0}, \tag{12.120}$$

where $\hat{\kappa}$ is defined by (12.119) with the minimizing value of $\boldsymbol{\beta}_2$. From (12.119), we see that the expression

$$\boldsymbol{\gamma}^{\top}\boldsymbol{Y}_*^{\top}(\boldsymbol{M}_{\boldsymbol{Z}} - \hat{\kappa}\boldsymbol{M}_{\boldsymbol{W}})\boldsymbol{Y}_*\boldsymbol{\gamma} = \boldsymbol{y}^{\top}(\boldsymbol{M}_{\boldsymbol{Z}} - \hat{\kappa}\boldsymbol{M}_{\boldsymbol{W}})\boldsymbol{Y}_*\boldsymbol{\gamma} - \boldsymbol{\beta}_2^{\top}\boldsymbol{Y}^{\top}(\boldsymbol{M}_{\boldsymbol{Z}} - \hat{\kappa}\boldsymbol{M}_{\boldsymbol{W}})\boldsymbol{Y}_*\boldsymbol{\gamma}$$

is equal to zero. In fact, both terms on the right-hand side here are zero, since, by (12.120), the second one vanishes. Therefore,

$$\boldsymbol{\gamma}^{\top}\boldsymbol{Y}_*^{\top}(\boldsymbol{M}_{\boldsymbol{Z}} - \hat{\kappa}\boldsymbol{M}_{\boldsymbol{W}})\boldsymbol{Y}_*\boldsymbol{\gamma} = 0.$$

If we premultiply this equation by $(\boldsymbol{Y}_*^{\top}\boldsymbol{M}_{\boldsymbol{W}}\boldsymbol{Y}_*)^{-1/2}$ and insert that factor multiplied by its inverse before $\boldsymbol{\gamma}$, we see, after some rearrangement, that

$$\big((\boldsymbol{Y}_*^{\top}\boldsymbol{M}_{\boldsymbol{W}}\boldsymbol{Y}_*)^{-1/2}\boldsymbol{Y}_*^{\top}\boldsymbol{M}_{\boldsymbol{Z}}\boldsymbol{Y}_*(\boldsymbol{Y}_*^{\top}\boldsymbol{M}_{\boldsymbol{W}}\boldsymbol{Y}_*)^{-1/2} - \hat{\kappa}\mathbf{I}\big)\boldsymbol{\gamma}^* = \boldsymbol{0},$$

where $\boldsymbol{\gamma}^* \equiv (\boldsymbol{Y}_*^{\top}\boldsymbol{M}_{\boldsymbol{W}}\boldsymbol{Y}_*)^{1/2}\boldsymbol{\gamma}$. This set of first-order conditions now has the form of a standard eigenvalue-eigenvector problem for a real symmetric matrix; see equation (12.115). Thus it is clear that $\hat{\kappa}$ is an eigenvalue of the matrix

$$(\boldsymbol{Y}_*^{\top}\boldsymbol{M}_{\boldsymbol{W}}\boldsymbol{Y}_*)^{-1/2}\boldsymbol{Y}_*^{\top}\boldsymbol{M}_{\boldsymbol{Z}}\boldsymbol{Y}_*(\boldsymbol{Y}_*^{\top}\boldsymbol{M}_{\boldsymbol{W}}\boldsymbol{Y}_*)^{-1/2}, \tag{12.121}$$

which depends only on observable data, and not on unknown parameters. In fact, $\hat{\kappa}$ must be the smallest eigenvalue, because it is the smallest possible value of the ratio (12.119). Given $\hat{\kappa}$, we can use equations (12.95) to compute the LIML estimates. It is worthy of note that, if there is only one endogenous variable in the matrix $\boldsymbol{Y}$, then the determinantal equation that determines the eigenvalues of (12.121) is just a quadratic equation, of which the smaller root is $\hat{\kappa}$, which can therefore be expressed in this case as a closed-form function of the data.