the purposes of the first of these, we need to assume that the zero functions f_t are continuously differentiable in the neighborhood of $\boldsymbol{\theta}_0$. If we perform a first-order Taylor expansion of $n^{1/2}$ times (9.59) around $\boldsymbol{\theta}_0$ and introduce some appropriate factors of powers of n, we obtain the result that

$$n^{-1/2}\boldsymbol{Z}^\top\boldsymbol{f}(\boldsymbol{\theta}_0) + n^{-1}\boldsymbol{Z}^\top\boldsymbol{F}(\bar{\boldsymbol{\theta}})n^{1/2}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) = \mathbf{0}, \tag{9.62}$$

where the $n \times k$ matrix $\boldsymbol{F}(\boldsymbol{\theta})$ has typical element

$$F_{ti}(\boldsymbol{\theta}) \equiv \frac{\partial f_t(\boldsymbol{\theta})}{\partial \theta_i}, \tag{9.63}$$

where θ_i is the i^{th} element of $\boldsymbol{\theta}$. This matrix, like $\boldsymbol{f}(\boldsymbol{\theta})$ itself, depends implicitly on the vector $\boldsymbol{y}$ and is therefore stochastic. The notation $\boldsymbol{F}(\bar{\boldsymbol{\theta}})$ in (9.62) is the convenient shorthand we introduced in Section 6.2: Row t of the matrix is the corresponding row of $\boldsymbol{F}(\boldsymbol{\theta})$ evaluated at $\boldsymbol{\theta} = \bar{\boldsymbol{\theta}}_t$, where the $\bar{\boldsymbol{\theta}}_t$ all satisfy the inequality

$$\|\bar{\boldsymbol{\theta}}_t - \boldsymbol{\theta}_0\| \leq \|\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\|.$$

The consistency of $\hat{\boldsymbol{\theta}}$ then implies that the $\bar{\boldsymbol{\theta}}_t$ also tend to $\boldsymbol{\theta}_0$ as $n \to \infty$.

The consistency of the $\bar{\boldsymbol{\theta}}_t$ implies that

$$\operatorname*{plim}_{n\to\infty} \frac{1}{n}\boldsymbol{Z}^\top\boldsymbol{F}(\bar{\boldsymbol{\theta}}) = \operatorname*{plim}_{n\to\infty} \frac{1}{n}\boldsymbol{Z}^\top\boldsymbol{F}(\boldsymbol{\theta}_0). \tag{9.64}$$

Under reasonable regularity conditions, we can apply a law of large numbers to the right-hand side of (9.64), and the probability limit is then deterministic. For asymptotic normality, we also require that it should be nonsingular. This is a condition of **strong asymptotic identification**, of the sort used in Section 6.2. By a first-order Taylor expansion of $\boldsymbol{\alpha}(\boldsymbol{\theta};\mu)$ around $\boldsymbol{\theta}_0$, where it is equal to $\mathbf{0}$, we see from the definition (9.60) that

$$\boldsymbol{\alpha}(\boldsymbol{\theta};\mu) \overset{a}{=} \operatorname*{plim}_{n\to\infty} \frac{1}{n}\boldsymbol{Z}^\top\boldsymbol{F}(\boldsymbol{\theta}_0)(\boldsymbol{\theta} - \boldsymbol{\theta}_0). \tag{9.65}$$

Therefore, the condition that the right-hand side of (9.64) is nonsingular is a strengthening of the condition that $\boldsymbol{\theta}$ is asymptotically identified. Because it is nonsingular, the system of equations

$$\operatorname*{plim}_{n\to\infty} \frac{1}{n}\boldsymbol{Z}^\top\boldsymbol{F}(\boldsymbol{\theta}_0)(\boldsymbol{\theta} - \boldsymbol{\theta}_0) = \mathbf{0}$$

has no solution other than $\boldsymbol{\theta} = \boldsymbol{\theta}_0$. By (9.65), this implies that $\boldsymbol{\alpha}(\boldsymbol{\theta};\mu) \neq \mathbf{0}$ for all $\boldsymbol{\theta} \neq \boldsymbol{\theta}_0$, which is the asymptotic identification condition.

Applying the results just discussed to equation (9.62), we find that

$$n^{1/2}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \overset{a}{=} -\Big(\operatorname*{plim}_{n\to\infty} \frac{1}{n}\boldsymbol{Z}^\top\boldsymbol{F}(\boldsymbol{\theta}_0)\Big)^{-1} n^{-1/2}\boldsymbol{Z}^\top\boldsymbol{f}(\boldsymbol{\theta}_0). \tag{9.66}$$