The regressand is an n–vector of 1s, and the regressor matrix is the matrix of contributions to the gradient, with typical element defined by (10.26). The artificial regression corresponds to the model implicitly defined by the matrix $\boldsymbol{G}(\boldsymbol{\theta})$, together with the ML estimator $\hat{\boldsymbol{\theta}}$. Let $\boldsymbol{m}(\boldsymbol{\theta})$ be the n–vector with typical element the moment function $m_t(y_t, \boldsymbol{\theta})$ that is to be tested, where once more the notation hides the dependence on the data. Then the testing regression is simplicity itself: We add $\boldsymbol{m}(\boldsymbol{\theta})$ to regression (15.23) as an extra regressor, obtaining

$$\boldsymbol{\iota} = \boldsymbol{G}(\boldsymbol{\theta})\boldsymbol{b} + c\boldsymbol{m}(\boldsymbol{\theta}) + \text{residuals}. \tag{15.24}$$

The test statistic is the t statistic on the extra regressor. The regressors here can be evaluated at any root-n consistent estimator, but it is most common to use the MLE $\hat{\boldsymbol{\theta}}$.

If several moment conditions are to be tested simultaneously, then we can form the $n \times r$ matrix $\boldsymbol{M}(\boldsymbol{\theta})$, each column of which is a vector of moment functions. The testing regression is then

$$\boldsymbol{\iota} = \boldsymbol{G}(\boldsymbol{\theta})\boldsymbol{b} + \boldsymbol{M}(\boldsymbol{\theta})\boldsymbol{c} + \text{residuals}. \tag{15.25}$$

When the regressors are evaluated at the MLE $\hat{\boldsymbol{\theta}}$, several asymptotically valid test statistics are available, including the explained sum of squares, n times the uncentered R^2, and the F statistic for the artificial hypothesis that $\boldsymbol{c} = \boldsymbol{0}$. The first two of these statistics are distributed asymptotically as $\chi^2(r)$ under the null hypothesis, as is r times the third. If the regressors in equation (15.25) are not evaluated at $\hat{\boldsymbol{\theta}}$, but at some other root-n consistent estimate, then only the F statistic is asymptotically valid.

The artificial regression (15.23) is valid for a very wide variety of models. Condition R2 requires that we be able to apply a central limit theorem to the scalar product $n^{-1/2}\boldsymbol{m}^\top(\boldsymbol{\theta}_0)\boldsymbol{\iota}$, where, as usual, $\boldsymbol{\theta}_0$ is the true parameter vector. If the expectation of each moment function $m_t(\boldsymbol{\theta}_0)$ is zero conditional on an appropriate information set Ω_t, then it is normally a routine matter to find a suitable central limit theorem. Condition R3 is also satisfied under very mild regularity conditions. What it requires is that the derivatives of $n^{-1}\boldsymbol{m}^\top(\boldsymbol{\theta})\boldsymbol{\iota}$ with respect to the elements of $\boldsymbol{\theta}$, evaluated at $\boldsymbol{\theta}_0$, should be given by minus the elements of the vector $n^{-1}\boldsymbol{m}^\top(\boldsymbol{\theta}_0)\boldsymbol{G}(\boldsymbol{\theta}_0)$, up to a term of order $n^{-1/2}$. Formally, we require that

$$\frac{1}{n}\sum_{t=1}^{n} \frac{\partial m_t(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\bigg|_{\boldsymbol{\theta}=\boldsymbol{\theta}_0} = -\frac{1}{n}\sum_{t=1}^{n} m_t(\boldsymbol{\theta}_0)\boldsymbol{G}_t(\boldsymbol{\theta}_0) + O_p(n^{-1/2}), \tag{15.26}$$

where $\boldsymbol{G}_t(\boldsymbol{\theta})$ is the t^{th} row of $\boldsymbol{G}(\boldsymbol{\theta})$. Readers are invited in Exercise 15.6 to show that equation (15.26) holds under the usual regularity conditions for ML estimation. This property and its use in conditional moment tests