for which the covariance matrix (13.09) of three consecutive observations has elements specified by equations (13.08). Show that necessary conditions for stationarity are that ρ_1 and ρ_2 lie inside the stationarity triangle which is shown in Figure 13.1 and defined by the inequalities

$$\rho_1 + \rho_2 < 1, \quad \rho_2 - \rho_1 < 1, \text{ and } \rho_2 > -1.$$

This can be done by showing that, outside the stationarity triangle, the matrix (13.09) is not positive definite.

⋆13.4 Show that, along the edges $\rho_1 + \rho_2 = 1$ and $\rho_1 - \rho_2 = -1$ of the AR(2) stationarity triangle, both roots of the polynomial $1 - \rho_1 z - \rho_2 z^2$ are real, one of them equal to 1 and the other greater than 1 in absolute value. Show further that, along the edge $\rho_2 = -1$, both roots are complex and equal to 1 in absolute value. How are these facts related to the general condition for the stationarity of an AR process?

13.5 Let $A(z)$ and $B(z)$ be two formal infinite power series in z, as follows:

$$A(z) = \sum_{i=0}^{\infty} a_i z^i \quad \text{and} \quad B(z) = \sum_{j=0}^{\infty} b_j z^j.$$

Let the formal product $A(z)B(z)$ be expressed similarly as the infinite series

$$C(z) = \sum_{k=0}^{\infty} c_k z^k.$$

Show that the coefficients c_k are given by the **convolution** of the coefficients a_i and b_j, according to the formula

$$c_k = \sum_{i=0}^{k} a_i b_{k-i}, \quad k = 0, 1, \ldots.$$

⋆13.6 Show that the method illustrated in Section 13.2 for obtaining the autocovariances of an ARMA(1, 1) process can be extended to the ARMA(p, q) case. Since explicit formulas are hard to obtain for general p and q, it is enough to indicate a recursive method for obtaining the solution.

13.7 Plot the autocorrelation function for the ARMA(2, 1) process

$$u_t = \rho_1 u_{t-1} + \rho_2 u_{t-2} + \varepsilon_t + \alpha_1 \varepsilon_{t-1}$$

for lags $j = 0, 1, \ldots, 20$ and for parameter values $\rho_1 = 0.8$, $\rho_2 = -0.6$, and $\alpha_1 = 0.5$. Repeat the exercise with $\alpha_1 = 0$, the other two parameters being unchanged, in order to see how the moving-average component affects the ACF in this case.

13.8 Consider the p Yule-Walker equations (13.95) for an AR(p) process as a set of simultaneous linear equations for the ρ_i, $i = 1, \ldots, p$, given the autocovariances v_i, $i = 0, 1, \ldots, p$. Show that the ρ_i which solve these equations